Quantum Field Theory Exercises week 13

Exercise 20: contractions and Wick's theorem for fermions

For Dirac fields the definitions of time ordering and normal ordering are extended in such a way that a minus sign is picked up for each interchange of fermionic operators. For example:

$$T\big(\hat\psi_{a_1}(x_1)\hat\psi_{a_2}(x_2)\hat\psi_{a_3}(x_3)\hat\psi_{a_4}(x_4)\big) \;=\; (-1)^3\,\hat\psi_{a_3}(x_3)\hat\psi_{a_1}(x_1)\hat\psi_{a_4}(x_4)\hat\psi_{a_2}(x_2) \quad \text{if} \quad x_3^0 > x_1^0 > x_4^0 > x_2^0$$

and

$$N(\hat a^s_{\vec p}\,\hat a^r_{\vec q}\,\hat a^{t\,\dagger}_{\vec l}) \;=\; (-1)^2\,\hat a^{t\,\dagger}_{\vec l}\,\hat a^s_{\vec p}\,\hat a^r_{\vec q} \;=\; (-1)^3\,\hat a^{t\,\dagger}_{\vec l}\,\hat a^r_{\vec q}\,\hat a^s_{\vec p}\ ,$$

where $a_1\,,\ldots,a_4$ are spinor indices. Based on these generalizations of time ordering and normal ordering we can extend the definition of the <u>contraction</u> of free Dirac fields:

$$\overbracket{\hat\psi_a(x)\hat{\bar\psi}_b}(y) \;=\; -\,\overbracket{\hat{\bar\psi}_b(y)\,\hat\psi_a}(x) \;\equiv\; \begin{cases} \{\hat\psi_a^+(x),\hat{\bar\psi}_b^-(y)\} & \text{if } \ x^0 > y^0 \\ \\ -\,\{\hat{\bar\psi}_b^+(y),\hat\psi_a^-(x)\} & \text{if } \ x^0 < y^0 \end{cases} \;=\; \big[S_F(x-y)\big]_{ab}\ \hat 1\ ,$$

where the superscript "+" refers to the positive-frequency part of the field and "−" to the negative-frequency part. The function $S_F(x-y)$ is the usual Feynman propagator of the Dirac theory. Furthermore

$$\overbracket{\hat\psi_a(x)\hat\psi_b}(y) \;=\; \overbracket{\hat{\bar\psi}_a(x)\hat{\bar\psi}_b}(y) \;=\; 0\ .$$

With this definition, the time-ordered expression can be rewritten as

$$T\big(\hat\psi_a(x)\hat{\bar\psi}_b(y)\big) \;=\; N\big(\hat\psi_a(x)\hat{\bar\psi}_b(y)\big) \;+\; \overbracket{\hat\psi_a(x)\hat{\bar\psi}_b}(y)\ .$$

<u>Wick's theorem</u> for free fields then states:

$$T\big(\hat\psi_{a_1}(x_1)\cdots\hat\psi_{a_n}(x_n)\big) \;=\; N\big(\hat\psi_{a_1}(x_1)\cdots\hat\psi_{a_n}(x_n)\ + \text{all possible contractions}\big)\ .$$

Browse through the steps on pages 37−39 of the lecture notes to convince yourself that the above statements are correct and work out $\langle 0|T\big(\hat\psi_{a_1}(x_1)\hat\psi_{a_2}(x_2)\hat{\bar\psi}_{a_3}(x_3)\hat{\bar\psi}_{a_4}(x_4)\big)|0\rangle$ in terms of Feynman propagators.

Exercise 21: an exam-style exercise about a Yukawa-like fermionic theory!!!

Make sure that you get started with this important exercise during the exercise class of week 13 and that you complete it the week after.

Consider the Lagrangian of the following fermionic theory:

$$\mathcal{L}(x) \;=\; \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m_\psi)\psi(x) \,+\, \frac{1}{2}\left[\partial_\mu\phi(x)\right]\left[\partial^\mu\phi(x)\right] \,-\, \frac{1}{2}m_\phi^2\phi^2(x) \,-\, g\bar{\psi}(x)\Gamma\psi(x)\phi(x)\ ,$$

where $\psi(x)$ is a Dirac field, $\bar{\psi}(x)$ its adjoint, $\phi(x)$ a real scalar field and Γ a 4×4 matrix in spinor space (i.e. $\Gamma = I_4$ for a scalar interaction and $\Gamma = i\gamma^5$ for a pseudo scalar interaction). The constant m_ψ represents the mass of the Dirac fermions and m_ϕ the mass of the scalar bosons. This Lagrangian contains an interaction between Dirac fermions and scalar bosons:

$$\mathcal{L}_{\rm int}(x) \;=\; -\,g\bar{\psi}(x)\Gamma\psi(x)\phi(x)\ .$$

(a) Derive the equations of motion of the theory.

(b) Determine the dimension of the various fields in the theory and explain why the coupling constant g must be dimensionless.

(c) Give a simple argument why in the non-interacting (free) quantized theory

$$\langle 0|T\big(\hat{\psi}_{a_I}(x)\,\hat{\psi}_{b_I}(y)\big)\,|0\rangle \;=\; \langle 0|T\big(\hat{\bar{\psi}}_{a_I}(x)\,\hat{\bar{\psi}}_{b_I}(y)\big)\,|0\rangle \;=\; 0\ ,$$

where the subscript I indicates that the fields are considered in the interaction picture. The indices a and b are spinor indices.

(d) Use this to determine

$$\langle 0|T\Big(\hat{\psi}_{a_I}(x_1)\,\hat{\bar{\psi}}_{b_I}(x_2)\,\hat{\phi}_I(x_3)\,{\rm e}^{-{\rm i}\int {\rm d}^4 z\ \hat{\mathcal{H}}_{{\rm int}_{\rm I}}({\rm z})}\Big)\,|0\rangle$$

to first order in the coupling constant g. Draw the corresponding position-space Feynman diagrams and express them in terms of

$$\langle 0|T\big(\hat{\psi}_{a_I}(x)\,\hat{\bar{\psi}}_{b_I}(y)\big)\,|0\rangle \;=\; \big[S_F(x-y)\big]_{ab} \;=\; -\,\langle 0|T\big(\hat{\bar{\psi}}_{b_I}(y)\,\hat{\psi}_{a_I}(x)\big)\,|0\rangle\ ,$$

$$\langle 0|T\big(\hat{\phi}_I(x)\,\hat{\phi}_I(y)\big)\,|0\rangle \;=\; D_F(x-y)\ ,$$

using solid lines to indicate the fermions and dashed ones to indicate the scalar bosons.

Mind the arrows and use explicit Dirac spinor labels in $\hat{\mathcal{H}}_{{\rm int}_I}(z)$ during intermediate steps to figure out how the various spinors and matrices should be contracted! To convince yourself that you got the right result, please check whether your findings are compatible with the Feynman rules for the Yukawa theory.

(e) Use the Feynman rules for the Yukawa theory to calculate the lowest-order matrix element for the process

$$\bar{\psi}(k_A,s_A)\,\bar{\psi}(k_B,s_B) \;\to\; \bar{\psi}(p_1,r_1)\,\bar{\psi}(p_2,r_2)\ ,$$

where $k_A,\cdots,p_2$ are the momenta of the incoming and outgoing ψ-antifermions and $s_A,\cdots,r_2$ the corresponding spin quantum numbers.

(f) Use the language of contractions to determine the relative signs of the contributions in part (e) and give a quantum mechanical explanation of your findings.

(g) Consider the one-loop self-energy for a scalar boson with arbitrary momentum p in the pure scalar version of the Yukawa theory (i.e. for $\Gamma = I_4$). Draw the corresponding diagram(s) and use γ-matrix properties to show that the self-energy is given by

$$-\,i\,\Sigma_\phi(p^2) \;\overset{\text{one-loop}}{=\!=\!=}\; -\frac{g^2}{4\pi^4}\int \mathrm{d}^4\ell_1\,\frac{m_\psi^2+\ell_1^2+\ell_1\cdot p}{[\ell_1^2-m_\psi^2+i\epsilon][(\ell_1+p)^2-m_\psi^2+i\epsilon]}\,.$$

(h) Consider an arbitrary 1-particle irreducible amputated loop diagram in the Yukawa theory with

N_F external fermion lines and N_B external boson lines,

P_F fermion propagators and P_B boson propagators,

V vertices and L loop momenta.

- Argue that $2V = N_F + 2P_F$, $V = N_B + 2P_B$ and $L = P_F + P_B - V + 1$.
- The superficial degree of divergence D of the diagram is obtained by treating all loop momenta and all components of the loop momenta as being of the same order of magnitude. Assume that these loop momenta are 4-dimensional and derive that $D = 4 - N_B - 3N_F/2$.

  Hint: first work out how the different types of particles contribute to D.
- Is the Yukawa theory renormalizable or not?

(i) Extra challenge: when the pieces of the puzzle do not seem to fit!

- Draw all one-loop superficially divergent diagrams.
- Indicate what counterterms you can use to cancel the divergence of each of these diagrams. Do you notice something strange?
- How could you resolve this issue?