# Exercises for Quantum Mechanics 3
# Set 13 (module 1)

**Exercise 27**: **Charge conjugation and the Dirac equation**

*The hidden description of opposite charge in the Dirac theory*

Consider the Dirac equation for spin-1/2 particles with charge $q$ and rest mass $m$ that are subject to a classical electromagnetic field:

$$\big(i\hbar\partial\!\!\!/ - mc - qA\!\!\!/(x)\big)\,\psi(x) \;=\; 0 \;, \tag{1}$$

where $A^\mu(x)$ is the real electromagnetic 4-vector potential. Use that

$$(\gamma^\mu)^* \;=\; \begin{cases} \gamma^\mu & \text{if } \mu = 0,1,3 \\ -\,\gamma^\mu & \text{if } \mu = 2 \end{cases} \qquad \text{and hence} \qquad \gamma^2(\gamma^\mu)^* \;=\; -\,\gamma^\mu\gamma^2$$

in the Dirac-representation to answer the following questions.

(i) Show that the charge-conjugated Dirac spinor $\psi_{C}(x) \equiv \gamma^2\psi^*(x)$ satisfies the same field equation, but with opposite charge:

$$\big(i\hbar\partial\!\!\!/ - mc + qA\!\!\!/(x)\big)\,\psi_{C}(x) \;=\; 0 \;.$$

Note: $\psi^*(x)$ implies complex conjugation, not hermitian conjugation.

(ii) What happens to $\psi(x)$ if we would perform charge conjugation twice?

(iii) Assume the electromagnetic field to be static, i.e. $A^\mu(x) = A^\mu(\vec{r}\,)$, and take

$$\psi^{(-E)}(x) \;=\; \exp(iEt/\hbar)\begin{pmatrix} \psi_1(\vec{r}\,) \\ \psi_2(\vec{r}\,) \\ \psi_3(\vec{r}\,) \\ \psi_4(\vec{r}\,) \end{pmatrix} \;\equiv\; \exp(iEt/\hbar)\begin{pmatrix} \psi_{U}(\vec{r}\,) \\ \psi_{D}(\vec{r}\,) \end{pmatrix}$$

to be a stationary solution to equation (1) with negative energy $-E$.

- What happens to the energy of this solution under charge conjugation?
- How does charge conjugation affect the 2-dimensional components $\psi_{U}$ and $\psi_{D}$?
- Argue that the expectation value of the Dirac spin operator $\hat{\vec{S}}$ changes its sign under the transition from $\psi^{(-E)}(x)$ to $\psi_{C}^{(-E)}(x)$.
  Hint: you may use that $\sigma^2\sigma^k\sigma^2 \;=\; -(\sigma^k)^*$ for $k = 1,2,3$.

*By means of charge conjugation a negative-energy state can be turned into a positive-energy state with opposite charge and opposite spin expectation value.*

**Exercise 28: The electromagnetic wave equation: relativity principle and spin**

Consider the free electromagnetic wave equation

$$\Box A^\mu(x) - \partial^\mu\big(\partial_\nu A^\nu(x)\big) = 0 ,$$

where $A^\mu(x)$ is the real electromagnetic 4-vector potential. The transformation characteristic of this 4-vector potential under Lorentz transformations is defined as follows (see the generic approach outlined on p. 105 of the lecture notes):

$$A'^{\,\mu}(x') \equiv G^\mu{}_\nu(\Lambda)\, A^\nu(x) .$$

The real invertible tensor $G^\mu{}_\nu(\Lambda)$ denotes how the Lorentz transformations act on the 4-dimensional space of intrinsic degrees of freedom spanned by the components of the 4-vector potential.

(i) Demonstrate that the relativity principle applies if $G^\mu{}_\nu(\Lambda) = \Lambda^\mu{}_\nu$, which implies that the 4-vector potential transforms as a contravariant vector field.

Remark: a second possible solution is $G^\mu{}_\nu(\Lambda) = \det(\Lambda)\, \Lambda^\mu{}_\nu$. The associated type of 4-vector potential is referred to as a pseudovector field or axial vector field.

(ii) Perform a spatial rotation by an infinitesimal angle $\delta\alpha$ about the axis oriented along the unit vector $\vec{e}_n$. In exercise 25 it has been shown that the spatial components of the position 4-vector $x^\mu$ transform as follows under this infinitesimal rotation:

$$x'^{\,\mu} = x^\mu + \delta\omega^\mu{}_\nu\, x^\nu \quad , \quad \delta\omega^\mu{}_\nu = \begin{cases} 0 & \text{if } \mu = 0 \,\vee\, \nu = 0 \\ -\,\delta\alpha \sum\limits_{l=1}^{3} \epsilon^{jkl}\, e_n^l & \text{if } \mu = j \,\wedge\, \nu = k \end{cases}$$

with $j,k = 1,2,3$ and $\epsilon^{jkl}$ as defined in equation (274) of the lecture notes.
Derive from this that the transformation characteristic of $A^\mu(x)$ under infinitesimal rotations is given by

$$A'^{\,\mu}(x') = \begin{cases} A^0(x) & \text{for } \mu = 0 \\ \sum\limits_{k=1}^{3} \Big( I - \dfrac{i}{\hbar}\, \delta\alpha\, \vec{e}_n \cdot \hat{\vec{S}} \Big)^{jk} A^k(x) & \text{for } \mu = j = 1,2,3 \end{cases} ,$$

where $\big(\hat{S}^l\big)^{jk} \equiv -\,i\hbar\, \epsilon^{ljk}$ acts as a 3×3 matrix on the vector potential $\vec{A}(x)$.

(iii) Finally, use the identity $\sum\limits_{l,k=1}^{3} \epsilon^{lkj}\, \epsilon^{lkj'} = 2\delta^{jj'}$ to prove that from a quantum mechanical point of view $A^\mu(x)$ can be associated with the description of spin-1 particles.