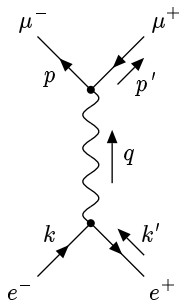
Quantum Field Theory Exercises week 14

Exercise 22: calculating with QED + an example of a Ward identity

One can learn the calculational tricks of the trade best by studying the QED reaction $e^-e^+ \to \mu^-\mu^+$ at lowest order in perturbation theory. Suppressing spin quantum numbers, the associated s-channel Feynman diagram and amplitude are given by



$$
\begin{aligned}
& i\mathcal{M}\big(e^-(k)e^+(k') \to \mu^-(p)\mu^+(p')\big) \\
&= \bar{v}(k')\left(+i|e|\gamma^\rho\right)u(k)\left(\frac{-ig_{\rho\sigma}}{q^2+i\epsilon}\right)\bar{u}(p)\left(+i|e|\gamma^\sigma\right)v(p') \\
&= \frac{ie^2}{q^2}\left[\bar{v}(k')\gamma^\rho u(k)\right]\left[\bar{u}(p)\gamma_\rho v(p')\right] ,
\end{aligned}
$$

where we have used that the charge of the electron and muon are equal to $-|e|$. For determining the (differential) cross section we need to calculate $|\mathcal{M}|^2$ as in § 4.3.

(a) First show that $\left[\bar{v}(k')\gamma^\nu u(k)\right]^* = \bar{u}(k)\gamma^\nu v(k')$ and $\left[\bar{u}(p)\gamma_\nu v(p')\right]^* = \bar{v}(p')\gamma_\nu u(p)$.

(b) We can make use of this to rewrite the expression for $|\mathcal{M}|^2$ in terms of traces in spinor space:

$$
|\mathcal{M}|^2 = \frac{e^4}{q^4}\,\mathrm{Tr}\Big(\left[v(k')\bar{v}(k')\right]\gamma^\rho\left[u(k)\bar{u}(k)\right]\gamma^\nu\Big)\,\mathrm{Tr}\Big(\left[u(p)\bar{u}(p)\right]\gamma_\rho\left[v(p')\bar{v}(p')\right]\gamma_\nu\Big) .
$$

If we are not able to produce polarized beams or to measure the polarization of the final-state muons, then we have to average over the initial-state polarizations and to sum over the final-state polarizations:

$$
\begin{aligned}
\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_{r,r'} |\mathcal{M}(s,s' \to r,r')|^2 &= \frac{1}{4}\sum_{\text{pol.}} |\mathcal{M}|^2 \\
&= \frac{e^4}{4q^4}\,\mathrm{Tr}\Big(\big[\sum_{s'} v^{s'}(k')\bar{v}^{s'}(k')\big]\gamma^\rho\big[\sum_s u^s(k)\bar{u}^s(k)\big]\gamma^\nu\Big) \times \\
&\quad \times\ \mathrm{Tr}\Big(\big[\sum_r u^r(p)\bar{u}^r(p)\big]\gamma_\rho\big[\sum_{r'} v^{r'}(p')\bar{v}^{r'}(p')\big]\gamma_\nu\Big) \\
&= \frac{e^4}{4q^4}\,\mathrm{Tr}\big([\not{k}' - m_e]\gamma^\rho[\not{k} + m_e]\gamma^\nu\big)\,\mathrm{Tr}\big([\not{p} + m_\mu]\gamma_\rho[\not{p}' - m_\mu]\gamma_\nu\big) .
\end{aligned}
$$

Apply the trace technology worked out in exercise 16 to obtain

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{pol.}} |\mathcal{M}|^2 &= \frac{4e^4}{q^4}\left[k'^\rho k^\nu + k'^\nu k^\rho - g^{\rho\nu}(k\cdot k' + m_e^2)\right]\left[p'_\nu p_\rho + p'_\rho p_\nu - g_{\rho\nu}(p\cdot p' + m_\mu^2)\right] \\
&= \frac{8e^4}{q^4}\left[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) + m_e^2(p\cdot p') + m_\mu^2(k\cdot k') + 2m_e^2 m_\mu^2\right] .
\end{aligned}
$$

(c) Use CM kinematics (see p. 62) and neglect m_e to simplify this to

$$\frac{1}{4}\sum_{\text{pol.}} |\mathcal{M}|^2 \;=\; e^4\Big[1+\frac{m_\mu^2}{E^2}+\big(1-\frac{m_\mu^2}{E^2}\big)\cos^2\theta\Big]\;,$$

where $E = E_{\text{CM}}/2$ is the energy of all four particles in the initial and final state.

Hint: the initial-state momenta are given by $k_A = k$ and $k_B = k'$, whereas the final-state momenta are given by $p_1 = p$ and $p_2 = p'$.

(d) Derive that the unpolarized total cross section is given by

$$\sigma_{\text{tot}}^{\text{unpol.}} \;=\; \Theta(E-m_\mu)\,\frac{\pi\alpha^2}{3E^2}\sqrt{1-m_\mu^2/E^2}\,\Big[1+\frac{m_\mu^2}{2E^2}\Big]\;,$$

where $\alpha = e^2/(4\pi)$ is the electromagnetic fine structure constant. Sketch the behaviour of this energy-dependent function.

(e) Attach an additional photon with momentum w and Minkowski index τ to the electron–positron line in the above Feynman diagram. Show that the corresponding matrix element $\mathcal{M}^\tau$ vanishes upon contraction with the photon momentum w_τ, i.e. $w_\tau\mathcal{M}^\tau = 0$.