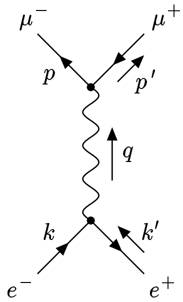
Quantum Field Theory Exercises week 14

Exercise 22: calculating with QED + an example of a Ward identity

One can learn the calculational tricks of the trade best by studying the QED reaction $e^-e^+ \to \mu^-\mu^+$ at lowest order in perturbation theory. Suppressing spin quantum numbers, the associated s-channel Feynman diagram and amplitude are given by



$$i\mathcal{M}\big(e^-(k)e^+(k') \to \mu^-(p)\mu^+(p')\big)$$

$$= \bar{v}(k')\big(+i|e|\gamma^\rho\big)u(k)\Big(\frac{-ig_{\rho\sigma}}{q^2+i\epsilon}\Big)\bar{u}(p)\big(+i|e|\gamma^\sigma\big)v(p')$$

$$= \frac{ie^2}{q^2}\big[\bar{v}(k')\gamma^\rho u(k)\big]\big[\bar{u}(p)\gamma_\rho v(p')\big]\ ,$$

where we have used that the charge of the electron and muon are equal to $-|e|$. For determining the (differential) cross section we need to calculate $|\mathcal{M}|^2$ as in § 4.3.

(a) First show that $\big[\bar{v}(k')\gamma^\nu u(k)\big]^* = \bar{u}(k)\gamma^\nu v(k')$ and $\big[\bar{u}(p)\gamma_\nu v(p')\big]^* = \bar{v}(p')\gamma_\nu u(p)$.

(b) We can make use of this to rewrite the expression for $|\mathcal{M}|^2$ in terms of traces in spinor space:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\,\mathrm{Tr}\Big(\big[v(k')\bar{v}(k')\big]\gamma^\rho\big[u(k)\bar{u}(k)\big]\gamma^\nu\Big)\,\mathrm{Tr}\Big(\big[u(p)\bar{u}(p)\big]\gamma_\rho\big[v(p')\bar{v}(p')\big]\gamma_\nu\Big)\ .$$

If we are not able to produce polarized beams or to measure the polarization of the final-state muons, then we have to average over the initial-state polarizations and to sum over the final-state polarizations:

$$\frac{1}{2}\sum_s\frac{1}{2}\sum_{s'}\sum_{r,r'}|\mathcal{M}(s,s'\to r,r')|^2 = \frac{1}{4}\sum_{\text{pol.}}|\mathcal{M}|^2$$

$$= \frac{e^4}{4q^4}\,\mathrm{Tr}\Big(\big[\sum_{s'}v^{s'}(k')\bar{v}^{s'}(k')\big]\gamma^\rho\big[\sum_s u^s(k)\bar{u}^s(k)\big]\gamma^\nu\Big)\times$$

$$\times\ \mathrm{Tr}\Big(\big[\sum_r u^r(p)\bar{u}^r(p)\big]\gamma_\rho\big[\sum_{r'}v^{r'}(p')\bar{v}^{r'}(p')\big]\gamma_\nu\Big)$$

$$= \frac{e^4}{4q^4}\,\mathrm{Tr}\big([\not{k}'-m_e]\gamma^\rho[\not{k}+m_e]\gamma^\nu\big)\,\mathrm{Tr}\big([\not{p}+m_\mu]\gamma_\rho[\not{p}'-m_\mu]\gamma_\nu\big)\ .$$

Apply the trace technology worked out in exercise 16 to obtain

$$\frac{1}{4}\sum_{\text{pol.}}|\mathcal{M}|^2 = \frac{4e^4}{q^4}\big[k'^\rho k^\nu + k'^\nu k^\rho - g^{\rho\nu}(k\cdot k' + m_e^2)\big]\big[p'_\nu p_\rho + p'_\rho p_\nu - g_{\rho\nu}(p\cdot p' + m_\mu^2)\big]$$

$$= \frac{8e^4}{q^4}\big[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) + m_e^2(p\cdot p') + m_\mu^2(k\cdot k') + 2m_e^2m_\mu^2\big]\ .$$

(c) Use CM kinematics (see p. 63) and neglect m_e to simplify this to

$$\frac{1}{4}\sum_{\text{pol.}} |\mathcal{M}|^2 \;=\; e^4\left[1+\frac{m_\mu^2}{E^2}+\big(1-\frac{m_\mu^2}{E^2}\big)\cos^2\theta\right],$$

where $E = E_{\rm CM}/2$ is the energy of all four particles in the initial and final state.
Hint: the initial-state momenta are given by $k_A = k$ and $k_B = k'$, whereas the final-state momenta are given by $p_1 = p$ and $p_2 = p'$.

(d) Derive that the unpolarized total cross section is given by

$$\sigma_{\rm tot}^{\rm unpol.} \;=\; \Theta(E-m_\mu)\,\frac{\pi\alpha^2}{3E^2}\,\sqrt{1-m_\mu^2/E^2}\,\left[1+\frac{m_\mu^2}{2E^2}\right],$$

where $\alpha = e^2/(4\pi)$ is the electromagnetic fine structure constant. Sketch the behaviour of this energy-dependent function.

(e) Attach an additional photon with momentum w and Minkowski index τ to the electron–positron line in the above Feynman diagram. Show that the corresponding matrix element $\mathcal{M}^\tau$ vanishes upon contraction with the photon momentum w_τ, i.e. $w_\tau\mathcal{M}^\tau \;=\; 0$.