Exercises for Quantum Mechanics 3
Set 14 (module 1)

Exercise 29: **Quantized electromagnetic field inside an enclosure**
Second quantization and the role of the energy spectrum

Consider the quantized electromagnetic field inside a finite cubic enclosure with volume V and periodic boundary conditions. The corresponding quantized vector potential reads

$$\hat{\vec{A}}(\vec{r},t) \;=\; \frac{1}{\sqrt{\epsilon_0}} \sum_{\vec{k}} \sum_{\lambda=1}^{2} \sqrt{\frac{\hbar}{2\omega_k}}\, \vec{u}_{\vec{k},\lambda}(\vec{r}) \left[\hat{a}_{\vec{k},\lambda}(t) \,+\, \eta_\lambda\, \hat{a}^{\dagger}_{-\vec{k},\lambda}(t) \right]$$

and the quantized versions of the electric and magnetic fields follow from the usual relations

$$\hat{\vec{\mathcal{E}}}(\vec{r},t) \;=\; -\frac{\partial}{\partial t}\,\hat{\vec{A}}(\vec{r},t) \qquad \text{and} \qquad \hat{\vec{\mathcal{B}}}(\vec{r},t) \;=\; \vec{\nabla} \times \hat{\vec{A}}(\vec{r},t)\ .$$

The definitions of the various quantities can be found in the lecture notes.

(i) Use appendix E of the lecture notes to prove the following operator identity for the total Hamilton operator of the quantized electromagnetic field:

$$\hat{H} \;\equiv\; \frac{\epsilon_0}{2} \int_V \mathrm{d}\vec{r} \left[\hat{\vec{\mathcal{E}}}(\vec{r},t) \cdot \hat{\vec{\mathcal{E}}}^{\,\dagger}(\vec{r},t) + c^2 \hat{\vec{\mathcal{B}}}(\vec{r},t) \cdot \hat{\vec{\mathcal{B}}}^{\,\dagger}(\vec{r},t) \right]$$
$$=\; \sum_{\vec{k}} \sum_{\lambda=1}^{2} \frac{1}{2}\,\hbar\omega_k \left[\hat{a}_{\vec{k},\lambda}(t)\, \hat{a}^{\dagger}_{\vec{k},\lambda}(t) + \hat{a}^{\dagger}_{-\vec{k},\lambda}(t)\, \hat{a}_{-\vec{k},\lambda}(t) \right] .$$

(ii) Show that this Hamilton operator has an energy spectrum that is bounded from below if the operators $\hat{a}^{\dagger}_{\vec{k},\lambda}(t)$ and $\hat{a}_{\vec{k},\lambda}(t)$ satisfy bosonic commutation relations

$$\left[\hat{a}_{\vec{k},\lambda}(t), \hat{a}^{\dagger}_{\vec{k}',\lambda'}(t) \right] \;=\; \delta_{\lambda,\lambda'}\, \delta_{\vec{k},\vec{k}'}\, \hat{1}\ .$$

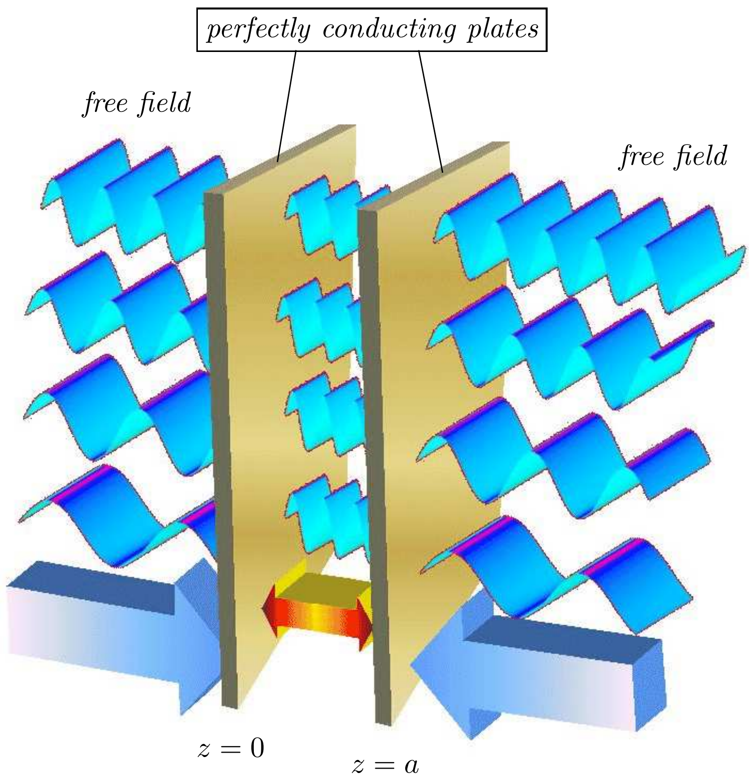
This is an instance of the so-called spin − statistics theorem, which states that spin and statistics are related in the relativistic QM. The quantization of relativistic wave equations belonging to arbitrary spins will result in the generic statement that bosons should have integer spin and fermions half integer spin. The first step in the prove of this theorem is to demand that the energy spectrum is bounded from below.

Exercise 30: **The Casimir effect (idealized two-plate configuration)**

An observable manifestation of vacuum energy

Consider a quantized electromagnetic field between two parallel, perfectly conducting plates. The plates are positioned at a distance a of each other and have a surface area S. Use a coordinate system with the z-axis perpendicular to the plates, such that the plates are positioned at $z = 0$ and $z = a$. Vectors parallel to the plates are denoted by the subscript "$\parallel$", such as $\vec{r}_\parallel$ for the coordinates and $\vec{k}_\parallel$ for the wave vectors. Assume the surface area S to be large enough that we may use the continuum limit for the quantization of the electromagnetic field parallel to the plates, i.e. for $\vec{k}_\parallel$. As known from classical electrodynamics, the requirement of perfect conductivity translates into the following conditions on the $\vec{\mathcal{E}}$- and $\vec{\mathcal{B}}$-fields:

$$\vec{\mathcal{E}}_\parallel(\vec{r}_\parallel, z=0, t) \;=\; \vec{\mathcal{E}}_\parallel(\vec{r}_\parallel, z=a, t) \;=\; \vec{0} \quad \text{and} \quad \mathcal{B}_z(\vec{r}_\parallel, z=0, t) \;=\; \mathcal{B}_z(\vec{r}_\parallel, z=a, t) \;=\; 0\,.$$



(i) Employ the Coulomb gauge and show that the boundary conditions

$$\vec{A}_\parallel(\vec{r}_\parallel, z=0, t) \;=\; \vec{A}_\parallel(\vec{r}_\parallel, z=a, t) \;=\; \vec{0}$$

for the vector potential yield the conditions for perfect conductivity.

(ii) Argue that between the plates the vector potential $\vec{A}(\vec{r}_\parallel, z, t)$ can thus be decomposed into periodic Fourier modes in the z-coordinate with period $2a$. This implies that the vector potential between the plates coincides with the vector potential originating from a periodic electromagnetic field with period $2a$ in the z-direction.

(iii) Now consider such a periodic electromagnetic field with period $2a$ in the z-direction. Then quantization occurs in the z-direction, with corresponding integer quantum number ν, and in the other two directions we may take the continuum limit.

Determine for such an electromagnetic field the quantum mechanical zero-point energy inside a volume $2Sa$. Hint: use § 4.2.4 of the lecture notes.

$$\text{Answer:} \quad E_0(2Sa) \;=\; \frac{\hbar c S}{4\pi} \int_0^\infty \mathrm{d}\vec{k}_{\parallel}^{\,2} \sum_{\nu=-\infty}^{\infty} \sqrt{\vec{k}_{\parallel}^{2} + (\nu\pi/a)^{\,2}}\ .$$

(iv) How would this zero-point energy change if we would be allowed to use the continuum limit in the z-direction as well? In that case we are dealing with the zero-point energy $E_0^{\rm free}(2Sa)$ of a free (unconstrained) electromagnetic field inside a volume $2Sa$.

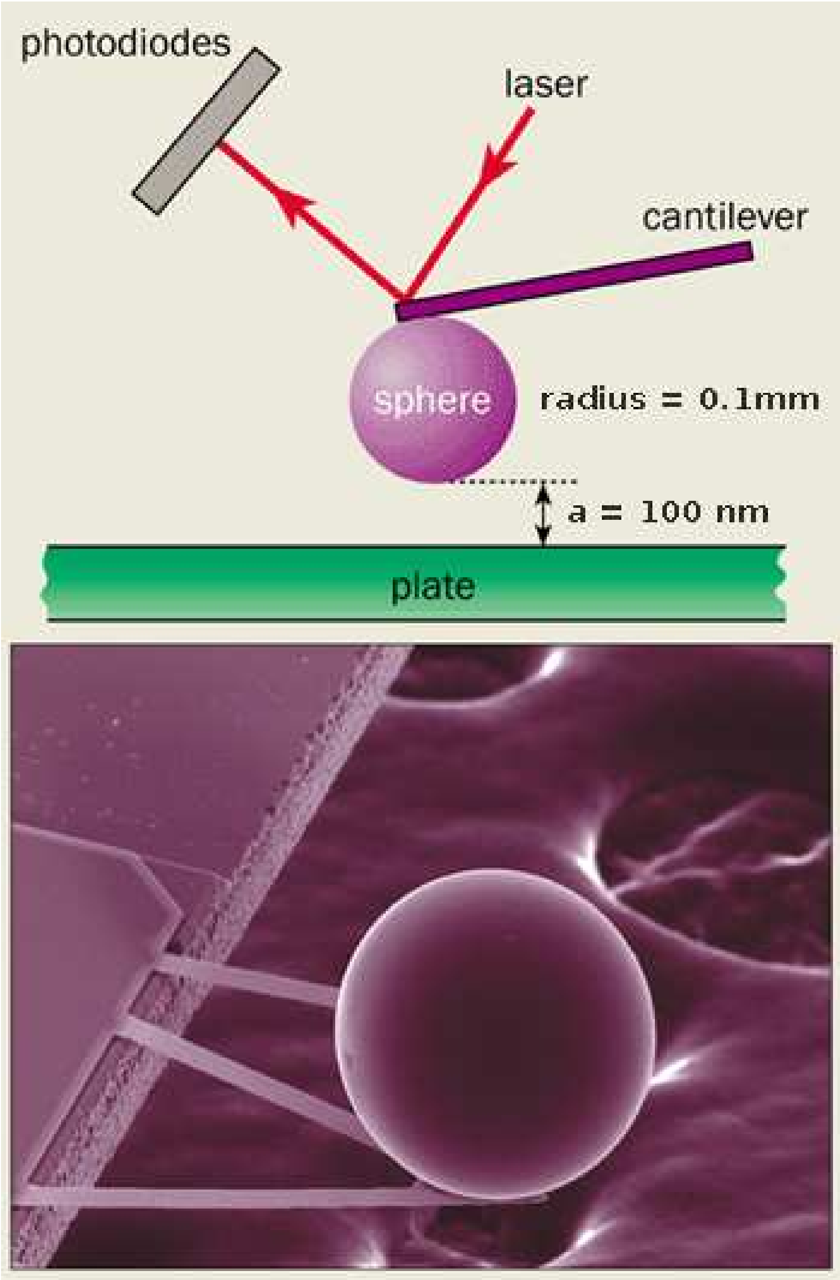
Since the actual volume between the plates is Sa, the zero-point energies in parts (iii) and (iv) should be halved for the system under consideration. Both expressions are (ultraviolet) divergent due to the absence of any restrictions on the high-energy range. As in quantum mechanics only energy differences are measurable, we are allowed to use the universal expression for the free electromagnetic field as absolute energy scale with respect to which all energies can be defined. This is called renormalization. Since two infinite expressions are being subtracted, this has to be done with due care (more information on this can be found in the bachelor thesis by Matthias Sars). At the end of the day one obtains the following expression for the renormalized zero-point energy belonging to the electromagnetic field between the two plates:

$$E_0^{\rm ren}(a) \;=\; -\,\frac{\pi^2}{720}\,\frac{\hbar c S}{a^3}\ .$$

(v) Determine the corresponding attractive force between the plates.

This purely quantum mechanical phenomenon was predicted by the Dutch physicist Hendrik Casimir in 1948 and is therefore referred to as the Casimir effect.

(vi) Divide the result of part (v) by the large plate area S to obtain the pressure on the plates (i.e. the force per unit area). Use dimensional arguments to reason that there actually is only one way to combine the finite system-specific quantities $\hbar, c$ and a into a pressure.



Experimentally it proved very difficult to align two plates with uniform micrometre precision. Eventually it took 50 years to successfully verify the Casimir effect experimentally. This was done by switching from a two-plate configuration to a sphere-and-plate one: S.K. Lamoreaux covered in this way the $a > 0.6\,\mu m$ *range in 1997 [official publication: Phys. Rev. Lett. 78, 5 (6 January 1997)], and U. Mohideen and A. Roy (see figure) covered the* $a > 0.1\,\mu m$ *range in 1998 [official publication: Phys. Rev. Lett. 81, 4549 (23 November 1998)].*