Exercises for Quantum Mechanics 3
Set 15 (module 1)

Exercise 31: **Thermal equilibrium between radiation and an atomic heat bath**
Deriving the key identity for obtaining Planck's law of black-body radiation

Consider a heat bath consisting of localized spin-0 atoms, such as spin-0 atoms in a solid-state environment. For convenience we assume the atoms to have two non-degenerate energy eigenstates $|\psi_A\rangle$ and $|\psi_B\rangle$ with energies E_A and $E_B = E_A - \hbar\omega$ $(\omega > 0)$. Assume this atomic heat bath to be in thermal equilibrium with electromagnetic radiation at temperature T, causing the reversible 1-photon process $A \rightleftarrows B+\gamma$ to be in equilibrium. The (average) equilibrium occupation of the atomic states A and B are denoted by $N(A)$ and $N(B)$, respectively.

(i) Derive from the equilibrium condition for the reversible 1-photon process that

$$\frac{N(B)}{N(A)} = \frac{W_{A\to B+\gamma}}{W_{B+\gamma\to A}} = \exp(\hbar\omega/k_{\rm B}T) \; ,$$

with $W_{A\to B+\gamma}$ and $W_{B+\gamma\to A}$ being the transition probabilities per unit of time per atom in the indicated initial state.

Hint: Maxwell–Boltzmann statistics may be used for the localized atoms.

(ii) Employ the operator identity

$$\big[F(\hat{\vec{r}}_j), \hat{\vec{p}}_j\big] = i\hbar\,\frac{\partial F}{\partial \hat{\vec{r}}_j}$$

to show that

$$\exp(i\vec{k}\cdot\hat{\vec{r}}_j)\,\vec{\epsilon}_\lambda(\vec{e}_k)\cdot\hat{\vec{p}}_j = \vec{\epsilon}_\lambda(\vec{e}_k)\cdot\hat{\vec{p}}_j\,\exp(i\vec{k}\cdot\hat{\vec{r}}_j)$$

for the position and momentum operators $\hat{\vec{r}}_j$ and $\hat{\vec{p}}_j$ of the $j^{\rm th}$ atom.

(iii) Deduce from this the detailed balancing condition for the $j^{\rm th}$ atom:

$$|\langle\psi_B^{(j)}|\exp(-i\vec{k}\cdot\hat{\vec{r}}_j)\,\vec{\epsilon}_\lambda(\vec{e}_k)\cdot\hat{\vec{p}}_j\,|\psi_A^{(j)}\rangle|^2 = |\langle\psi_A^{(j)}|\exp(i\vec{k}\cdot\hat{\vec{r}}_j)\,\vec{\epsilon}_\lambda(\vec{e}_k)\cdot\hat{\vec{p}}_j\,|\psi_B^{(j)}\rangle|^2 \; .$$

(iv) Assume the atoms in the heat bath to have on average no preferred direction, allowing photons to be emitted/absorbed without directional preference. This implies that all photon states with quantum numbers λ and $\{\vec{k}\,:\; k = \omega/c\}$ share the same occupation number $\bar{n}$ in the radiation field. Use the detailed-balancing condition of part (iii) to argue that the following should hold at first order in perturbation theory:

$$j^{\rm th} \text{ atom}: \quad \frac{W_{A^{(j)}\to B^{(j)}+\gamma_{\vec{k},\lambda}}}{W_{B^{(j)}+\gamma_{\vec{k},\lambda}\to A^{(j)}}} \approx \frac{\bar{n}+1}{\bar{n}} \quad \Rightarrow \quad \text{all atoms}: \quad \frac{W_{A\to B+\gamma}}{W_{B+\gamma\to A}} \approx \frac{\bar{n}+1}{\bar{n}} \; .$$

Hint: use equations (364)–(366) of the lecture notes and neglect for convenience's sake the magnetic spin interaction.

(v) Prove that the photons with quantum numbers λ and $\{\vec{k}: \; k=\omega/c\}$ are distributed according to

$$\bar{n} \; = \; \frac{1}{\exp(\hbar\omega/k_{\rm B}T)-1} \; .$$

Exercise 32: **Alternative particle interpretation in the free spin-1/2 theory**

The reason why the 1-particle Dirac theory was such a success

Consider the quantized solution $\hat{\psi}(x)$ to the Dirac equation for free spin-1/2 particles, as given in equation (398) of the lecture notes.

(i) Prove that this solution satisfies the following set of anticommutation relations:

$$\big\{\hat{\psi}_i(\vec{x},t),\hat{\psi}_{i'}(\vec{x}\,',t)\big\} \; = \; \big\{\hat{\psi}_i^\dagger(\vec{x},t),\hat{\psi}_{i'}^\dagger(\vec{x}\,',t)\big\} \; = \; 0 \; ,$$

$$\big\{\hat{\psi}_i(\vec{x},t),\hat{\psi}_{i'}^\dagger(\vec{x}\,',t)\big\} \; = \; \delta_{ii'}\,\delta(\vec{x}-\vec{x}\,')\,\hat{1} \qquad (i,i'=1,\cdots,4 \;\text{ spinor indices}) \; .$$

In this proof you should use the completeness relations for Dirac spinors and Fourier components:

$$\sum_\lambda \Big(u_\lambda^+(\vec{p})\,u_\lambda^+(\vec{p})^\dagger \; + \; u_\lambda^-(\vec{p})\,u_\lambda^-(\vec{p})^\dagger\Big) \; = \; \frac{E_{\vec{p}}}{mc^2}\,I_4 \; ,$$

$$\sum_{\vec{p}} u_{\vec{p}}(\vec{x}\,)\,u_{\vec{p}}^*(\vec{x}\,') \; = \; \frac{1}{V}\sum_{\vec{p}} \exp\big(i\vec{p}\cdot[\vec{x}-\vec{x}\,']/\hbar\big) \; = \; \delta(\vec{x}-\vec{x}\,') \; .$$

(ii) Why can one conclude from this that an alternative particle interpretation is hidden in the free spin-1/2 theory?

(iii) Finally, prove that $\langle 0_e|\hat{\psi}_i(\vec{x},t)\,\hat{\psi}_{i'}^\dagger(\vec{x}\,',t)|0_e\rangle = \delta_{ii'}\,\delta(\vec{x}-\vec{x}\,')$ for the electron vacuum $|0_e\rangle$ defined in equation (406) of the lecture notes.

Armed with this knowledge you shoud be able to understand § 5.2.1 of the lecture notes, where an explicit link is provided between the many-particle Dirac theory and the successful 1-particle theory.