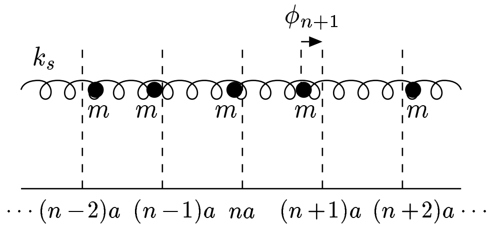
Quantum Field Theory Exercises week 1

Exercise 1: A 1-dimensional periodic lattice (going from discrete to continuum)

Consider a 1-dimensional periodic lattice of N point particles of mass m that are connected by springs with spring constant k_s. The equilibrium position of particle n is given by $\bar{x}_n = na$, where $a > 0$ is the fixed average inter-particle distance. The positions of the particles relative to the equilibrium positions are indicated by the coordinates $\phi_n(t)$, which are assumed to be small with respect to a. The periodicity condition implies that particle $N+1$ coincides with particle 1.



(a) Explain that the Lagrangian of the discrete system is given by:

$$L\big(\{\phi_n\},\{\dot{\phi}_n\}\big) \;=\; \sum_{n=1}^{N}\left(\frac{1}{2}m\dot{\phi}_n^2(t) - \frac{1}{2}k_s\left[\phi_{n+1}(t)-\phi_n(t)\right]^2\right).$$

(b) Derive the Lagrange equations for the coordinates $\phi_n(t)$.

(c) Considered on a macroscopic length scale $L = Na \gg a$ the lattice can be viewed as a continuous system, i.e. a can be viewed as being infinitesimal compared to L. The corresponding continuum limit is given by $\phi_n(t) \to \sqrt{a}\,\phi(x,t)|_{x=na}$, where $\phi(x,t) \in [0,L]$ is a smooth function describing the lattice fluctuations. Taylor expand in "a" up to the first non-vanishing order a^p $(p>0)$ to prove that:

- $L\big(\{\phi_n\},\{\dot{\phi}_n\}\big) \;\to\; \int_0^L dx\left(\frac{1}{2}m\dot{\phi}^2 - \frac{1}{2}k_s a^2\left[\partial\phi/\partial x\right]^2\right) \;\equiv\; \int_0^L dx\,\mathcal{L}(\phi,\partial_t\phi,\partial_x\phi)$,
- the Lagrange equations derived in (b) become $m\Big(\partial_t^2 - \dfrac{k_s a^2}{m}\partial_x^2\Big)\phi(x,t) = 0$,

where $\partial_t \equiv \partial/\partial t$ and $\partial_x \equiv \partial/\partial x$.

(d) Show that the latter equation of motion coincides with the Euler-Lagrange equation of the continuous system.

(e) What are the solutions of this equation going to describe?

(f) Determine the Hamiltonian density of the continuous system.

Exercise 2: Free electromagnetic theory (dealing with Minkowski indices)

Solve exercise 2.1(a) from Peskin and Schroeder. Additional hints:

- Use that

$$\epsilon^{ijk} \;=\; \begin{cases} +1 & \text{if } (i,j,k) = \text{even permutation of (1,2,3)} \\ -1 & \text{if } (i,j,k) = \text{odd permutation of (1,2,3)} \\ 0 & \text{else} \end{cases} ,$$

  so $\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}$ with summation over i and j implied.

- From this definition it follows that $(\vec{a}\times\vec{b})^k = \epsilon^{ijk}a^i b^j$ for the cross product (= vector product) of two three-dimensional vectors $\vec{a}$ and $\vec{b}$. Again summation over i and j is implied here.

- Two of the four Maxwell equations follow from the general form of $F_{\mu\nu}$, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, rather than from the equations of motion.