Exercises for Quantum Mechanics 3
Set 1

Aim: getting acquainted with creation and annihilation operators

Exercise 1: Basis states for bosonic many-particle systems

Consider the states

$$|n_1, n_2, \cdots\rangle \;=\; \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\,\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots|\Psi^{(0)}\rangle \;\equiv\; \prod_j \frac{(\hat{a}_j^\dagger)^{n_j}}{\sqrt{n_j!}}\,|\Psi^{(0)}\rangle\;,$$

$$|\Psi^{(0)}\rangle \;\equiv\; |0,0,\cdots\rangle\;, \qquad \text{with} \quad \langle\Psi^{(0)}|\Psi^{(0)}\rangle \;=\; 1 \qquad \text{and} \qquad \hat{a}_j|\Psi^{(0)}\rangle \;=\; 0\;,$$

where the creation and annihilation operators $\hat{a}_j^\dagger$ and $\hat{a}_j$ satisfy bosonic commutation relations

$$\big[\hat{a}_j^\dagger, \hat{a}_k^\dagger\big] \;=\; \big[\hat{a}_j, \hat{a}_k\big] \;=\; 0 \qquad \text{and} \qquad \big[\hat{a}_j, \hat{a}_k^\dagger\big] \;=\; \delta_{jk}\,\hat{1}\;.$$

Each occupation number n_j can take the values $0, 1, 2, \cdots$. It represents the number of particles in the fully specified 1-particle eigenstate belonging to the discrete eigenvalue q_j $(j = 1, 2, \cdots)$ of the complete set of 1-particle observables $\hat{q}$.

(i) Use equation (21) of the lecture notes to show that these commutation relations retain their form when switching to another 1-particle representation.

(ii) Prove by induction that

$$\big[\hat{a}_j, (\hat{a}_j^\dagger)^{n_j}\big] \;=\; n_j\,(\hat{a}_j^\dagger)^{n_j-1}\;.$$

(iii) Argue that in the definition of the above-given states the order of the creation operators does not matter.

(iv) Use part (ii) to derive the following relations:

$$\hat{a}_j|\cdots, n_j, \cdots\rangle \;=\; \sqrt{n_j}\;|\cdots, n_j{-}1, \cdots\rangle\;,$$

$$\hat{a}_j^\dagger|\cdots, n_j, \cdots\rangle \;=\; \sqrt{n_j{+}1}\;|\cdots, n_j{+}1, \cdots\rangle\;.$$

(v) – Prove that the above-given states form an orthonormal set.

Hint: consider to this end $\langle n_1', n_2', \cdots|n_1, n_2, \cdots\rangle$, write the state "bra" in terms of annihilation operators, and subsequently use the expression for $\hat{a}_j|\cdots, n_j, \cdots\rangle$ given in part (iv) as well as the vacuum property $\langle\Psi^{(0)}|\hat{a}_j^\dagger = 0$.

– The underlying basic principle at work is that as many $\hat{a}_j$ as $\hat{a}_j^\dagger$ operators should occur between $\langle\Psi^{(0)}|$ and $|\Psi^{(0)}\rangle$ in order for a vacuum expectation value to be non-vanishing. Why is that?

Exercise 2: **Basis states for fermionic many-particle systems**

Again consider the states

$$|n_1, n_2, \cdots\rangle \;=\; \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\,\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\,\cdots\,|\Psi^{(0)}\rangle \;\equiv\; \prod_j \frac{(\hat{a}_j^\dagger)^{n_j}}{\sqrt{n_j!}}\,|\Psi^{(0)}\rangle\ ,$$

$$|\Psi^{(0)}\rangle \;\equiv\; |0, 0, \cdots\rangle\ , \qquad \text{with} \quad \langle\Psi^{(0)}|\Psi^{(0)}\rangle \;=\; 1 \qquad \text{and} \qquad \hat{a}_j\,|\Psi^{(0)}\rangle \;=\; 0\ ,$$

where the creation and annihilation operators $\hat{a}_j^\dagger$ and $\hat{a}_j$ in this case satisfy fermionic anticommutation relations

$$\{\hat{a}_j^\dagger, \hat{a}_k^\dagger\} \;=\; \{\hat{a}_j, \hat{a}_k\} \;=\; 0 \qquad \text{and} \qquad \{\hat{a}_j, \hat{a}_k^\dagger\} \;=\; \delta_{jk}\,\hat{1}\ .$$

In accordance with Pauli's exclusion principle, the occupation numbers can only take the values 0 or 1. Again n_j represents the number of particles in the fully specified 1-particle eigenstate belonging to the discrete eigenvalue q_j $(j = 1, 2, \cdots)$ of the complete set of 1-particle observables $\hat{q}$.

(i) Show that these anticommutation relations retain their form when switching to another 1-particle representation.

(ii) Argue that in the definition of the above-given states the order of the creation operators does matter now.

(iii) Derive the following relations:

$$\hat{a}_j\,|\cdots, n_j, \cdots\rangle \;=\; \delta_{n_j,1}\,(-1)^{N_{<j}}\,|\cdots, n_j{-}1, \cdots\rangle\ ,$$

$$\hat{a}_j^\dagger\,|\cdots, n_j, \cdots\rangle \;=\; \delta_{n_j,0}\,(-1)^{N_{<j}}\,|\cdots, n_j{+}1, \cdots\rangle\ ,$$

with

$$N_{<j} \;\equiv\; \sum_{k=1}^{j-1} n_k\ .$$

(iv) Prove that the above-given states form an orthonormal set.

Hint: consider to this end $\langle n_1', n_2', \cdots | n_1, n_2, \cdots\rangle$, write the state "bra" in terms of annihilation operators, and subsequently use the expression for $\hat{a}_j\,|\cdots, n_j, \cdots\rangle$ given in part (iii) as well as the vacuum property $\langle\Psi^{(0)}|\,\hat{a}_j^\dagger \;=\; 0$.