# Quantum Field Theory Exercises week 2

**Exercise 2 (continued)**
Complete exercise 2.1 from Peskin and Schroeder. Don't forget to use the additional hints given in last week's exercises.

**Exercise 3: A free complex scalar doublet (example of extended gauge symmetries)**

Consider the free quantum field theory for two complex-valued scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the same mass. The Lagrangian of this theory is given by

$$\mathcal{L} \;=\; (\partial_\mu \phi_1^*)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^*)(\partial^\mu \phi_2) - m^2(\phi_1^* \phi_1 + \phi_2^* \phi_2).$$

This free quantum field theory resembles the one for a doublet of Higgs fields. It is easiest to analyze this theory by considering $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$ as basic dynamical variables, rather than the real and imaginary parts of $\phi_{1,2}(x)$.

(a) Find the equations of motion for $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$.

(b) Determine the conjugate momenta belonging to $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$, indicating these conjugate momenta by $\pi_{1,2}(x)$ and $\pi_{1,2}^*(x)$ respectively.

(c) Show that the Hamiltonian is given by

$$H \;=\; \int \mathrm{d}\vec{x}\, \Big[\, \pi_1^*\pi_1 + \pi_2^*\pi_2 + (\vec{\nabla}\phi_1^*)\cdot(\vec{\nabla}\phi_1) + (\vec{\nabla}\phi_2^*)\cdot(\vec{\nabla}\phi_2) + m^2(\phi_1^*\phi_1 + \phi_2^*\phi_2) \Big]\,.$$

(d) View the fields $\phi_{1,2}$ as being components of a vector $\vec{\Phi}$ and rewrite the Lagrangian in terms of this vector. Then show that the Lagrangian is invariant (i.e., does not change) under the continuous transformation

$$\vec{\Phi}(x) \;\to\; \exp(i\alpha)\,\vec{\Phi}(x) \qquad\quad (\alpha \in \mathbb{R} \text{ independent of } x)\;.$$

This is called a <u>global $U(1)$ transformation</u>, where global refers to the fact that $\alpha$ is independent of $x$ and $U(1)$ indicates that all fields are multiplied by a phase factor.

Determine the corresponding conserved Noether current and charge.

Hint: please keep a close eye on the order of the vectors, with row vectors having to precede column vectors in expressions that do not have open indices. This notion will serve you well at a later stage.

(e) Show that the Lagrangian is invariant under the continuous <u>global $SU(2)$-transformations</u>

$$\vec{\Phi}(x) \;\to\; \exp(i\alpha^k\sigma^k)\,\vec{\Phi}(x) \qquad\quad (\alpha^{1,2,3} \in \mathbb{R} \text{ independent of } x)\;,$$

with $\sigma^k$ the hermitian Pauli matrices and with summation over $k = 1,2,3$ implied.

Determine the corresponding three conserved Noether currents and charges.

Hint: this time also the matrices should act on the vectors from the correct side (i.e. from the left on column vectors and from the right on row vectors).