Exercises for Quantum Mechanics 3
Set 2

Exercise 3: **Basis of N-particle state functions in a continuous representation**

Aim: linking up with the approach in the lecture course Quantum Mechanics 2

Consider an identical N-particle system with $\hat{a}_j^\dagger$ and $\hat{a}_j$ being the creation and annihilation operators belonging to the discrete 1-particle basis $\{\,|q_j\rangle :\ j = 1, 2, \cdots\}$. As worked out in exercises 1 and 2, the orthonormal basis states

$$|n_1, n_2, \cdots\rangle \;=\; \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\,\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots|\Psi^{(0)}\rangle \qquad\qquad (n_1, n_2, \cdots \;=\; 0, 1, 2, \cdots)$$

span the Fock space for systems consisting of an arbitrary number of such identical particles. Since we are dealing here with a fixed number of particles, we have to restrict this set of states to those basis states for which the occupation numbers n_j add up to N. Such sets of occupation numbers we denote compactly by $\{n_j\}_N$.

(i) Take $\hat{1}_N$ to be the unit operator in the N-particle subspace. Argue that

$$\sum_{\{n_j\}_N} |n_1, n_2, \cdots\rangle\langle n_1, n_2, \cdots| \;=\; \hat{1}_N\ .$$

Subsequently we switch to a continuous representation. To this end we consider the creation and annihilation operators $\hat{a}^\dagger(k)$ and $\hat{a}(k)$ belonging to the continuous 1-particle basis $\{\,|k\rangle :\ k \in \text{continuous spectrum}\}$. We will prove now that also the N-particle states

$$\Big\{|k_1, \cdots, k_N\rangle \;\equiv\; \frac{1}{\sqrt{N!}}\,\hat{a}^\dagger(k_1)\cdots\hat{a}^\dagger(k_N)|\Psi^{(0)}\rangle :\quad k_1, \cdots, k_N \in \{k\}\Big\}$$

form an orthonormal N-particle basis.

(ii) First step: use the unitary basis transformation that connects both sets of creation and annihilation operators to derive that

$$\int\!\mathrm{d}k_1\cdots\!\int\!\mathrm{d}k_N\ |k_1, \cdots, k_N\rangle\langle k_1, \cdots, k_N| \;=\; \sum_{i_1=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\hat{a}_{i_1}^\dagger\cdots\hat{a}_{i_N}^\dagger|\Psi^{(0)}\rangle\langle\Psi^{(0)}|\hat{a}_{i_N}\cdots\hat{a}_{i_1}}{N!}\ .$$

(iii) Finally, prove that the following holds for both bosons and fermions:

$$\sum_{i_1=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\frac{\hat{a}_{i_1}^\dagger\cdots\hat{a}_{i_N}^\dagger|\Psi^{(0)}\rangle\langle\Psi^{(0)}|\hat{a}_{i_N}\cdots\hat{a}_{i_1}}{N!} \;=\; \sum_{\{n_j\}_N}|n_1, n_2, \cdots\rangle\langle n_1, n_2, \cdots| \;=\; \hat{1}_N\ .$$

Hint: on the left-hand side the creation and annihilation operators can occur multiple times. Collect these operators by bringing $\hat{a}_{i_1}^\dagger\cdots\hat{a}_{i_N}^\dagger$ in the form $(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\cdots$ Determine the correct weight factors by counting the number of ways in which the same set of occupation numbers can be realized by the quantum numbers $i_1, \cdots, i_N$.

(iv) An arbitrary N-particle state function $|\Psi\rangle$ can thus be represented in the k-representation by the function $\psi(k_1,\cdots,k_N) \equiv \langle k_1,\cdots,k_N|\Psi\rangle$. Explain that this function is totally symmetric for bosons and totally antisymmetric for fermions.

Exercise 4: Spatial pair interactions in Fock space

Aim: preparation for the discussion of superfluidity (see § 1.6.4)

Consider a system consisting of an arbitrary number of identical spin-s particles of mass m. The particles experience a mutual spatial pair interaction described by the observable $U(\hat{\vec{r}}_1 - \hat{\vec{r}}_2)$. This observable is diagonal in the position representation, leading to the following total operator for spatial pair interactions:

$$\hat{U}_{\text{pair}} \equiv \frac{1}{2}\sum_{\sigma_1,\sigma_2}\int d\vec{r}_1\int d\vec{r}_2\, U(\vec{r}_1-\vec{r}_2)\,\hat{\psi}^\dagger_{\sigma_1}(\vec{r}_1)\,\hat{\psi}^\dagger_{\sigma_2}(\vec{r}_2)\,\hat{\psi}_{\sigma_2}(\vec{r}_2)\,\hat{\psi}_{\sigma_1}(\vec{r}_1)\ , \tag{1}$$

where $\hat{\psi}^\dagger_\sigma(\vec{r})$ and $\hat{\psi}_\sigma(\vec{r})$ are the creation and annihilation operators in the position representation for particles with spin component $\sigma\hbar$ along the quantization axis.

(i) How do we call this type of many-particle observable?

(ii) Challenge: what are the properties of this many-particle observable if the particles have spin 1/2 and the potential is a 3-dimensional δ-function: $U(\vec{r}_1-\vec{r}_2) \propto \delta(\vec{r}_1-\vec{r}_2)$? What can you tell about the total spin of the interacting particle pairs?

(iii) Show that equation (1) takes the form

$$\hat{U}_{\text{pair}} = \frac{1}{2}\sum_{\sigma_1,\sigma_2}\int d\vec{k}\int d\vec{p}_1\int d\vec{p}_2\, \mathcal{U}(\vec{k})\,\hat{a}^\dagger_{\sigma_1}(\vec{p}_1)\,\hat{a}^\dagger_{\sigma_2}(\vec{p}_2)\,\hat{a}_{\sigma_2}(\vec{p}_2+\hbar\vec{k})\,\hat{a}_{\sigma_1}(\vec{p}_1-\hbar\vec{k})$$

in the momentum representation, with corresponding creation and annihilation operators $\hat{a}^\dagger_\sigma(\vec{p})$ and $\hat{a}_\sigma(\vec{p})$. Use the definitions and Fourier integrals given in the lecture notes as well as the decomposition $U(\vec{r}) \equiv \int d\vec{k}\,\mathcal{U}(\vec{k})\exp(i\vec{k}\cdot\vec{r})$. Moreover, in the derivation you will also need to use the integral identities (A.14) and (A.15).

(iv) Which symmetry property is hidden inside this expression?

(v) Argue that $\mathcal{U}(\vec{k}) \equiv (2\pi)^{-3}\int d\vec{r}\,U(\vec{r})\exp(-i\vec{k}\cdot\vec{r})$ will depend exclusively on $|\vec{k}\,|$ if $U(\vec{r})$ is a function of the distance $|\vec{r}\,|$ only.

Remark: if the system is confined to a macroscopic enclosure, the momentum spectrum will become discrete and the Fourier integrals will have to be replaced by corresponding Fourier series (see App. A and Ch. 2 of the lecture notes).