Quantum Field Theory Exercises week 3

Exercise 4: The complex KG theory (particle interpretation and "vacuum properties")

Consider the free quantum field theory for a complex-valued scalar field $\phi(x)$ with mass m. As derived in exercise 3, the Hamilton operator of this theory is given by

$$\hat{H} \;=\; \int \mathrm{d}\vec{x} \left[\hat{\pi}^\dagger(\vec{x}\,)\,\hat{\pi}(\vec{x}\,) + \vec{\nabla}\hat{\phi}^\dagger(\vec{x}\,)\cdot\vec{\nabla}\hat{\phi}(\vec{x}\,) + m^2\hat{\phi}^\dagger(\vec{x}\,)\,\hat{\phi}(\vec{x}\,) \right] .$$

(a) In order to diagonalize the Hamilton operator, introduce creation and annihilation operators by decomposing the Schrödinger-picture field $\hat{\phi}(\vec{x}\,)$ in spatial plane-wave modes. In chapter 2 of Peskin and Schroeder this is done for a hermitian Klein-Gordon field. You can reduce the problem at hand to this case by rewriting $\hat{\phi}(\vec{x}\,)$ as

$$\hat{\phi}(\vec{x}\,) \;=\; \frac{\hat{\phi}_1(\vec{x}\,) + i\,\hat{\phi}_2(\vec{x}\,)}{\sqrt{2}} \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}} + \hat{b}^\dagger_{-\vec{p}}\right) ,$$

with $\hat{\phi}_1(\vec{x}\,)$ and $\hat{\phi}_2(\vec{x}\,)$ commuting hermitian fields.

- Express $\hat{a}_{\vec{p}}$ and $\hat{b}^\dagger_{-\vec{p}}$ in terms of the creation and annihilation operators $\hat{a}^\dagger_{1,\vec{p}}\,, \hat{a}_{1,\vec{p}}$ and $\hat{a}^\dagger_{2,\vec{p}}\,, \hat{a}_{2,\vec{p}}$ of $\hat{\phi}_1(\vec{x}\,)$ and $\hat{\phi}_2(\vec{x}\,)$, respectively.
- Determine the commutation relations for $\hat{a}_{\vec{p}}\,$, $\hat{a}^\dagger_{\vec{q}}$, $\hat{b}_{\vec{k}}$ and $\hat{b}^\dagger_{\vec{l}}$.
  You can use the commutation relations for real scalar fields for this.
- Argue that

$$\hat{\pi}(\vec{x}\,) \;=\; \frac{\hat{\pi}_1(\vec{x}\,) - i\,\hat{\pi}_2(\vec{x}\,)}{\sqrt{2}} \;=\; -\,i\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{2}}\;e^{i\vec{p}\cdot\vec{x}}\left(\hat{b}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}}\right) .$$

  On the left-hand side of this equation you will have to figure out what the conjugate momentum of $i\hat{\phi}_2$ is in relation to the conjugate momentum $\hat{\pi}_2$ of $\hat{\phi}_2$.

(b) Show that $\hat{H}$ can be expressed in terms of the number operators $\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}$ and $\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}$. Also indicate the constant remainder proportional to the unit operator, which is usually referred to as the zero-point energy.

(c) Do the same for the "charge" operator $\hat{Q}$, which is given by

$$\hat{Q} \;=\; \int \mathrm{d}\vec{x}\;\hat{j}^0(\vec{x}\,) \;=\; -\,i\int \mathrm{d}\vec{x}\left[\hat{\phi}^\dagger(\vec{x}\,)\,\hat{\pi}^\dagger(\vec{x}\,) - \hat{\pi}(\vec{x}\,)\,\hat{\phi}(\vec{x}\,)\right]$$

up to a constant factor.

(d) What can you conclude on the basis of parts (b) and (c) about the particle interpretation of the complex Klein-Gordon theory, i.e. what is the energy and charge of the fundamental quanta that are created by $\hat{a}^\dagger_{\vec{p}}$ and $\hat{b}^\dagger_{\vec{p}}$?

(e) If we would have started from classical definitions of the Hamiltonian and charge in which the order of the fields was reversed, we would upon quantization have obtained the same zero-point energy but a zero-point charge with the opposite sign. Argue that this is indeed the case.

(f) Prove that the vacuum expectation value $\langle 0|\hat{\vec{j}}\,(\vec{x}\,)|0\rangle$ vanishes for the spatial part of the Noether (charge) current

$$\hat{\vec{j}}\,(\vec{x}\,) \;=\; -\,i\left(\vec{\nabla}\hat{\phi}^{\dagger}(\vec{x}\,)\right)\hat{\phi}(\vec{x}\,) + i\,\hat{\phi}^{\dagger}(\vec{x}\,)\left(\vec{\nabla}\hat{\phi}(\vec{x}\,)\right)\,.$$

(g) This was to be expected, since a non-zero current would give the vacuum a preferred direction. Actually this should hold for all inertial observers. Given that $\hat{j}^{0}(\vec{x}\,)$ and $\hat{\vec{j}}(\vec{x}\,)$ can be combined into the contravariant four-vector $\hat{j}^{\mu}(\vec{x}\,)$, reason that $\langle 0|\hat{j}^{0}(\vec{x}\,)|0\rangle$ better be zero.

(h) In most QFT textbooks normal ordering is used to remove unwanted (read: unphysical) vacuum properties. Another method that could be applied is Weyl-ordering, which involves averaging over all possible definitions (i.e. orderings) of the quantum observables.

Discuss the pros and cons of both methods by considering the implications for the energy and charge properties of the vacuum.