Exercises for Quantum Mechanics 3
Set 4

Exercise 6: Additional aspects of the forced oscillator

Consider the forced oscillator introduced in § 1.6.2 of the lecture notes as well as the set $\big\{ |\lambda\rangle \;=\; \sum\limits_{n=0}^{\infty} \exp(-|\lambda|^2/2)\, \frac{\lambda^n}{\sqrt{n!}}\, |n\rangle \;:\; \lambda \in \mathbb{C} \big\}$ of all coherent states, consisting of the set of quasi-particle vacua belonging to arbitrary forced oscillators. In this set the states $\big\{ |n\rangle \;:\; n = 0, 1, \cdots \big\}$ represent the usual orthonormal basis of energy eigenstates of the linear harmonic oscillator of § 1.6.1.

(i) Show that the overlap probability between two of these coherent states is given by $|\langle\lambda'|\lambda\rangle|^2 = \exp\big(-|\lambda|^2 - |\lambda'|^2 + \lambda\lambda'^* + \lambda^*\lambda'\big) \;=\; \exp\big(-|\lambda-\lambda'|^2\big)$.

(ii) The coherent states are eigenstates of the lowering operator $\hat{a}$ of the linear harmonic oscillator. Why does that not automatically imply that these eigenstates are mutually orthogonal?

(iii) Why does the raising operator $\hat{a}^\dagger$ not possess eigenstates, in contrast to $\hat{a}$?
Base your argumentation on a decomposition in terms of the energy eigenstates of the linear harmonic oscillator.

(iv) Coherent states corresponding to very large eigenvalues $|\lambda| \gg 1$ are often associated with "classical states". Explain why the result of part (i) supports such a point of view, given the properties that you would expect for classical states.

(v) Use the bosonic identity $\big[\hat{a}, F(\hat{a},\hat{a}^\dagger)\big] \;=\; \partial F(\hat{a},\hat{a}^\dagger)/\partial\hat{a}^\dagger$ to derive that

$$\hat{c} \;\equiv\; \hat{S}(\lambda)\,\hat{a}\,\hat{S}^\dagger(\lambda) \;=\; \hat{a} - \lambda\hat{1} \qquad \text{for} \qquad \hat{S}(\lambda) \;=\; \exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a}) \;.$$

Remark: the identity that you just used is the creation/annihilation analogue of the well-known 1-dimensional commutation relation $\big[\hat{x}, F(\hat{x},\hat{p})\big] \;=\; i\hbar\,\partial F(\hat{x},\hat{p})/\partial\hat{p}$.

The transition from the bosonic quanta belonging to the linear harmonic oscillator to the bosonic quasi quanta belonging to the forced oscillator can thus be represented by a unitary transformation.