Exercises for Quantum Mechanics 3
Set 5

Exercise 7: **Additional aspects of the example of weakly repulsive spin-0 bosons**

Use §1.6.4–1.6.6 of the lecture notes while working out this exercise

Consider the Bogolyubov transformation discussed in § 1.6.5 and define the normalized state

$$|\tilde{0}\rangle \;\equiv\; \frac{1}{u_1}\exp\Big(-\frac{v_1}{u_1}\,\hat{a}_1^\dagger\,\hat{a}_2^\dagger\Big)\,|0,0\rangle \;=\; \frac{1}{u_1}\sum_{n=0}^{\infty}\Big(\frac{-\,v_1}{u_1}\Big)^n\,|n,n\rangle\ ,$$

where $|n_1,n_2\rangle$ are the usual occupation-number basis states in the original particle interpretation (i.e. the particle interpretation prior to the Bogolyubov transformation). As can be read off straightforwardly this state consists of coherently created particle pairs!

(i) Show that $\hat{a}_1\,|\tilde{0}\rangle \;=\; -\frac{v_1}{u_1}\,\hat{a}_2^\dagger\,|\tilde{0}\rangle$ and $\hat{a}_2\,|\tilde{0}\rangle \;=\; -\frac{v_1}{u_1}\,\hat{a}_1^\dagger\,|\tilde{0}\rangle$.
Hint: you will again need the bosonic identity $\big[\hat{a}, F(\hat{a},\hat{a}^\dagger)\big] \;=\; \partial F(\hat{a},\hat{a}^\dagger)/\partial\hat{a}^\dagger$.

(ii) – Use this to derive that the state $|\tilde{0}\rangle$ satisfies the identities.

$$\hat{c}_1\,|\tilde{0}\rangle \;=\; \hat{c}_2\,|\tilde{0}\rangle \;=\; 0 \quad , \quad \hat{c}_1^\dagger\,|\tilde{0}\rangle \;=\; \frac{1}{u_1}\,\hat{a}_1^\dagger\,|\tilde{0}\rangle \qquad \text{and} \qquad \hat{c}_2^\dagger\,|\tilde{0}\rangle \;=\; \frac{1}{u_1}\,\hat{a}_2^\dagger\,|\tilde{0}\rangle$$

for the quasi-particle creation and annihilation operators $\hat{c}_{1,2}^\dagger$ and $\hat{c}_{1,2}$ of § 1.6.5.

– How would you call the state $|\tilde{0}\rangle$?

(iii) Show that $\langle\tilde{0}|\,\hat{a}_1^\dagger\,\hat{a}_1\,|\tilde{0}\rangle = \langle\tilde{0}|\,\hat{a}_2^\dagger\,\hat{a}_2\,|\tilde{0}\rangle = v_1^2$.

In § 1.6.6 we performed for each pair of spin-0 particles with opposite momenta $\vec{q} \neq \vec{0}$ and $-\vec{q}$ a Bogolyubov transformation with transformation parameters $u_{\vec{q}}$ and $v_{\vec{q}}$ in order to switch from a description in terms of particles to a description in terms of quasi particles. These quasi particles correspond to an approximation of the actual excitations that can occur in the weakly repulsive system. The approximation for the ground state of the interacting system, i.e. the state without any excitations, then has the following form in terms of the non-interacting ground state $|n_{\vec{0}} = N, n_{\vec{k}\neq\vec{0}} = 0\rangle$:

$$|\Psi_{\text{ground}}\rangle \;\approx\; |\Psi_{\text{Bog}}\rangle \;\equiv\; \prod_{\text{pairs}}\frac{1}{u_{\vec{q}}}\exp\Big(-\frac{v_{\vec{q}}}{u_{\vec{q}}}\,\hat{b}_{\vec{q}}^\dagger\,\hat{b}_{-\vec{q}}^\dagger\Big)\,|n_{\vec{0}} = N, n_{\vec{k}\neq\vec{0}} = 0\rangle\ ,$$

where the product runs over all different pairs $(\vec{q}\,, -\vec{q}\,) \neq \vec{0}$. The definition of the operator $\hat{b}_{\vec{q}\neq\vec{0}}^\dagger$ can be found on page 37 of the lecture notes.

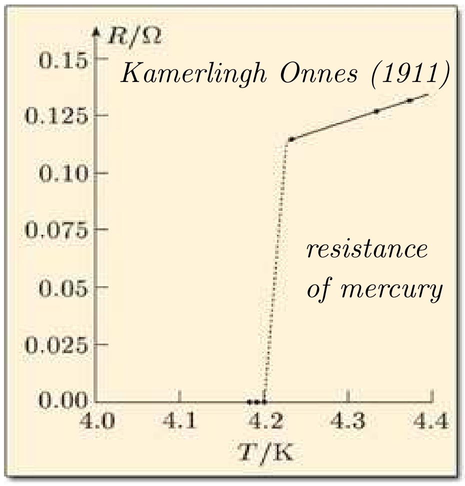
(iv) Explain that $|\Psi_{\text{Bog}}\rangle$ describes a state containing a fixed number N of spin-0 particles.

(v) – Approximate $\langle\Psi_{\text{Bog}}|\,\hat{n}_{\vec{0}}\,|\Psi_{\text{Bog}}\rangle \approx \langle\Psi_{\text{Bog}}|\big(\hat{N} - \sum_{\vec{k}\neq\vec{0}}\hat{b}_{\vec{k}}^\dagger\,\hat{b}_{\vec{k}}\big)|\Psi_{\text{Bog}}\rangle$ in terms of N and $v_{\vec{q}}$.

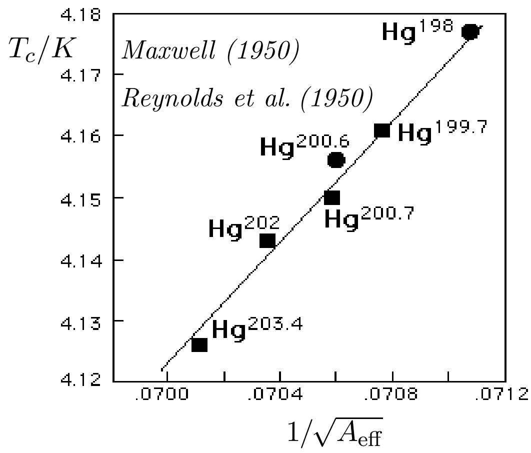
– What is the requirement on the parameters $v_{\vec{q}}$ (and thus on the interaction) in order for the approximation method in § 1.6.4 to make sense?

Cooper pairs in low-temperature superconductors

(background reading)

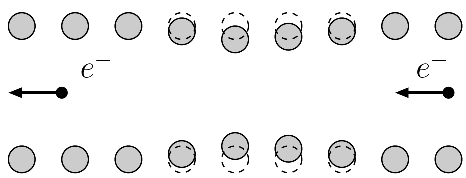


We speak of a superconductor if the resistance of a conductor vanishes abruptly below a certain critical temperature T_c (see figure). In that case part of the charge carrying conduction electrons is in a collective superfluid state, which allows these electrons to move through the ion lattice of the conductor without experiencing friction. The low speed of such a superfluid flow is compensated by the sheer numbers of the superfluid collective, making it nevertheless possible to generate very substantial currents.



The metal-ion lattice matters: by using different isotopes for the lattice ions, it was demonstrated experimentally that not only the electron configuration but also the metal-ion lattice itself plays a role in the phenomenon of superconductivity. It was observed that the critical temperature T_c for low-temperature superconductivity depends on the mass A_{eff} (in atomic mass units) of the isotope that was used for the lattice ions.

BCS theory and Cooper pairs: the physical picture behind the previous observations was formulated in 1957 in the so-called BCS (Bardeen−Cooper−Schrieffer) theory.



The Cooper-pair concept

The BCS theory states that a conduction electron that is propagating through a piece of metal excites vibrations (waves) in the metal-ion lattice (phonon emission). The lattice ions are positively charged and can thus be attracted by the negatively charged electron, causing the lattice to become slightly deformed. The time scale on which such a deformation evolves in time is very slow (adiabatic) compared to the motions of the much lighter electrons. Before the lattice has the time to rebound another electron can be attracted by the induced fluctuation in the charge density. This electron can subsequently absorb the quantized lattice vibration (phonon absorption). As a result of this long-distance interaction via the lattice it is possible to induce a net attractive interaction between pairs of electrons that would otherwise repel each other.

Such an attractive electron pair is called a Cooper pair. It is the "fluid" of these Cooper pairs that can flow through the lattice without friction at sufficiently low temperatures (in analogy with the two-fluid model for superfluid ^{4}He).

Normal conductors versus superconductors: based on this model we can instantly understand one of the fundamental differences between normal conductors and superconductors. The more rigid the metal-ion lattice is

- the better a normal conductor becomes, in view of the larger mean free path for the conduction electrons;
- the worse a superconductor becomes, since a diminished possibility to interact with the lattice has a negative impact on the formation of Cooper pairs.

Exercise 8: An attractive fermion pair above an occupied Fermi sea
This serves as an example for the Cooper instability in superconductors

Consider two identical fermions with mass m and momenta $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$ from the spectrum

$$\{\vec{p} = \hbar\vec{k} \,:\, k_{x,y,z} \;=\; 0, \pm 2\pi/L, \pm 4\pi/L, \cdots\} \qquad (L \in \mathbb{R}) \ .$$

Assume all 1-particle momentum states with $|\vec{k}| < k_F$ to be occupied by other fermions of the same type. For convenience these other fermions are left out from the discussion, except for the effective exclusion-principle requirement $|\vec{k}_{1,2}| \geq k_F$ on our fermion pair. Besides this all spin and identity aspects of the two particles are omitted in the discussion, i.e. you may treat the system as if it was consisting of two distinguishable spin-0 particles! In fact, the essence of the Cooper-instability phenomenon is already contained in this highly simplified system (see below).

Suppose both particles have an attractive mutual pair interaction with matrix elements

$$\langle \vec{k}_1', \vec{k}_2' | \hat{V} | \vec{k}_1, \vec{k}_2 \rangle \;=\; -\lambda\, \delta_b\, \delta_{b'}\, \delta_{\vec{k}_1+\vec{k}_2\,,\,\vec{k}_1'+\vec{k}_2'} \ ,$$

with λ, k_F and k_M positive real constants and k_M slightly larger than k_F. The orthonormal states $|\vec{k}_1, \vec{k}_2\rangle$ describe particle pairs with momentum eigenvalues $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$. The step functions δ_b and $\delta_{b'}$ mark a narrow momentum band just above the Fermi sea for which the pair interactions are attractive:

$$\delta_b \;=\; \begin{cases} 1 & k_F \leq |\vec{k}_{1,2}| \leq k_M \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \delta_{b'} \;=\; \begin{cases} 1 & k_F \leq |\vec{k}_{1,2}'| \leq k_M \\ 0 & \text{else} \end{cases} \ .$$

The task that we set ourselves now is to look for the ground state of the 2-particle system with Hamilton operator

$$\hat{H} \;=\; \frac{\hat{\vec{p}}_1^{\,2} + \hat{\vec{p}}_2^{\,2}}{2m} + \hat{V} \;.$$

Since the total momentum of the particle pair is conserved here, we consider for the ground state a linear combination of orthonormal pair states with fixed total momentum $\hbar\vec{Q}$ and variable relative momentum $\hbar\vec{q}$:

$$|\Psi_{\vec{Q}}\rangle \;\equiv\; \sum_{\vec{q}} \delta_b\, C_{\vec{Q}}(\vec{q})\, |\vec{k}_1 = \tfrac{1}{2}\vec{Q} + \vec{q}\,,\, \vec{k}_2 = \tfrac{1}{2}\vec{Q} - \vec{q}\,\rangle \;,$$

where $\vec{q}$ is restricted to a limited domain since $k_F \le |\tfrac{1}{2}\vec{Q} \pm \vec{q}\,| \le k_M$.

(i) Why does it not make sense to involve pair states outside the narrow interaction band $k_F \le |\tfrac{1}{2}\vec{Q} \pm \vec{q}\,| \le k_M$ while looking for the ground state?

(ii) Derive from the eigenvalue equation $\hat{H}\,|\Psi_{\vec{Q}}\rangle = E\,|\Psi_{\vec{Q}}\rangle$ that

$$C_{\vec{Q}}(\vec{q}\,') \;=\; \frac{-\lambda}{E - \hbar^2\vec{Q}^2/(4m) - \hbar^2\vec{q}\,'^2/m} \sum_{\vec{q}} \delta_b\, C_{\vec{Q}}(\vec{q}\,) \quad \text{for} \quad k_F \le |\tfrac{1}{2}\vec{Q} \pm \vec{q}\,'| \le k_M \;.$$

Hint: project on the orthonormal states $\langle \vec{k}_1' \equiv \tfrac{1}{2}\vec{Q} + \vec{q}\,',\, \vec{k}_2' \equiv \tfrac{1}{2}\vec{Q} - \vec{q}\,'|$.

(iii) Sum this to $F_{\vec{Q}}(E) \;\equiv\; \sum_{\vec{q}\,'} \dfrac{\delta_{b'}}{E - \hbar^2\vec{Q}^2/(4m) - \hbar^2\vec{q}\,'^2/m} \;=\; -\dfrac{1}{\lambda}\,.$

(iv) Presentation assignment: if you are not on the presentation team, you can treat this assignment as a bonus exercise.

- Make a qualitative sketch of $F_{\vec{Q}}(E)$ and argue that the lowest energy eigenvalue E_{ground} has the following property:

  $$E_{\text{ground}} \;<\; \hbar^2 k_F^2/m \;=\; \text{the lowest energy of the two separate particles,}$$

  irrespective of the size of $\lambda > 0$. To this end you should first explain how $F_{\vec{Q}}(E)$ roughly behaves while E passes a total kinetic energy eigenvalue.

  Consequence: there is a "bound" state, however weak the attractive interaction actually may be. This non-perturbative fermionic pairing phenomenon is referred to as the Cooper instability.

- In what sense does this differ from the customary binding that we, for instance, observe in molecules and what is the role that the Fermi sea plays in all this?
- Explain briefly what will happen if the particles in the Fermi sea are taken into account in the full analysis and we keep the interaction band fixed.