Exercises for Quantum Mechanics 3
Set 6

Exercise 9: Bogolyubov transformation for fermions

Aim: highlighting the special properties of fermionic quasi particles

Consider the fermionic version of the Bogolyubov transformation as described in § 1.7.2 of the lecture notes and let's address a few selected aspects that will feature in the discussions of superconductivity (module 2) and relativistic quantum mechanics (module 1).

(i) Consider the normalized quasi-particle vacuum state $|\tilde{0}\rangle$ for which $\hat{c}_{1,2}|\tilde{0}\rangle \equiv 0$.

- Show that $|\tilde{0}\rangle$ can be decomposed as follows (up to a phase factor) in terms of the original basis states $|n_1, n_2\rangle$:

$$|\tilde{0}\rangle \;=\; u_1|0,0\rangle - v_1|1,1\rangle \;=\; \big(u_1 - v_1\hat{a}_1^\dagger\hat{a}_2^\dagger\big)|0,0\rangle \ .$$

  Note of warning: pay due attention to relative minus signs.

- Demonstrate that for $u_1 \neq 0$ this can be recast in the form

$$|\tilde{0}\rangle \;=\; u_1\exp\Big(-\frac{v_1}{u_1}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\Big)|0,0\rangle \ .$$

- Determine the expectation values $\langle\tilde{0}|\hat{a}_1^\dagger\hat{a}_1|\tilde{0}\rangle$ and $\langle\tilde{0}|\hat{a}_2^\dagger\hat{a}_2|\tilde{0}\rangle$.

(ii) Next we have a look at a special Bogolyubov transformation with $u_1 = 1 - v_1 = 0$.

- What type of quasi particles are we dealing with in that case?
- Also these quasi particles are subject to the customary Pauli exclusion principle. What does this special quasi-particle exclusion principle actually imply?
- Deduce straight from the fundamental (anti)commutation relations that this special type of Bogolyubov transformation is possible for fermionic systems, but not for bosonic sytems.

(iii) Consider the additive 1-particle quantity belonging to the many-particle observable

$$\hat{A} \;=\; a\,\hat{a}_1^\dagger\hat{a}_1 + b\,\hat{a}_2^\dagger\hat{a}_2 \ ,$$

with a and b real 1-particle eigenvalues. As we can read off straightaway, in the original particle interpretation $\hat{a}_1^\dagger$ creates a particle with eigenvalue a and $\hat{a}_2^\dagger$ a particle with eigenvalue b. Subsequently we apply again the special Bogolyubov transformation with $u_1 = 1 - v_1 = 0$. What does the quantity described by $\hat{A}$ now have to say about the quasi particles?

Exercise 10: **Transition from a pure ensemble to a mixed ensemble**

Aim: getting used to the concept of "integrating out degrees of freedom"

Consider a decay process in which two different spin-0 particles are produced in a pure pair state with total momentum $\vec{P}$. Suppose that this pure state is given by

$$|\Psi\rangle \;=\; \int \mathrm{d}\vec{p}_1\, f(\vec{p}_1)\, \hat{a}^\dagger(\vec{p}_1)\, \hat{b}^\dagger(\vec{P}-\vec{p}_1)\,|0\rangle \quad \text{with} \quad f(\vec{p}_1) \in \mathbb{C} \quad \text{and} \quad \vec{P}, \vec{p}_1 \in \mathbb{R}^3 ,$$

where $f(\vec{p}_1) \neq 0$ if the process is allowed kinematically for the given momentum $\vec{p}_1$. The state $|0\rangle$ is the vacuum state and $\hat{a}^\dagger(\vec{p}_1)$ and $\hat{b}^\dagger(\vec{P}-\vec{p}_1)$ are the creation operators in the continuous momentum representation belonging to the first and second particle, respectively. Separately these operators satisfy the usual bosonic commutation relations for a continuous representation, i.e. for instance

$$\big[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{p}_1^{\,\prime})\big] \;=\; \delta(\vec{p}_1 - \vec{p}_1^{\,\prime})\,\hat{1} \qquad \text{and} \qquad \big[\hat{b}(\vec{p}_2), \hat{b}^\dagger(\vec{p}_2^{\,\prime})\big] \;=\; \delta(\vec{p}_2 - \vec{p}_2^{\,\prime})\,\hat{1} \;.$$

(i) Now consider an observable $\hat{A}$ that applies exclusively to the first particle. Argue that this implies that $\hat{b}(\vec{p}_2)$ commutes with $\hat{A}$ as well as $\hat{a}(\vec{p}_1)$ and $\hat{a}^\dagger(\vec{p}_1)$.

(ii) Take $\hat{a}^\dagger(\vec{p}_1)\,|0\rangle \equiv |\vec{p}_1\rangle$ to be the momentum eigenstate of particle 1 with eigenvalue $\vec{p}_1$ and use this to show that

$$\langle\Psi|\hat{A}|\Psi\rangle \;=\; \int \mathrm{d}\vec{p}_1\, |f(\vec{p}_1)|^2\, \langle 0|\,\hat{a}(\vec{p}_1)\,\hat{A}\,\hat{a}^\dagger(\vec{p}_1)\,|0\rangle \;=\; \int \mathrm{d}\vec{p}_1\, |f(\vec{p}_1)|^2\, \langle\vec{p}_1|\hat{A}\,|\vec{p}_1\rangle \;.$$

(iii) Demonstrate that this can be rewritten as $\langle\Psi|\hat{A}|\Psi\rangle = \mathrm{Tr}_1(\hat{A}\hat{\rho}_1)$, where

$$\hat{\rho}_1 \;\equiv\; \int \mathrm{d}\vec{p}_1\, |f(\vec{p}_1)|^2\, \hat{a}^\dagger(\vec{p}_1)\,|0\rangle\langle 0|\,\hat{a}(\vec{p}_1) \;=\; \int \mathrm{d}\vec{p}_1\, |f(\vec{p}_1)|^2\, |\vec{p}_1\rangle\langle\vec{p}_1|$$

describes a mixed ensemble characterized by the momentum eigenstates $|\vec{p}_1\rangle$ of particle 1 with statistical weights $W(\vec{p}_1) = |f(\vec{p}_1)|^2$. The trace Tr_1 refers to the complete state space for particle 1, which implies that we have to integrate over all momenta:

$$\mathrm{Tr}_1(\hat{O}) \;\equiv\; \int \mathrm{d}\vec{p}_1\, \langle\vec{p}_1|\hat{O}|\vec{p}_1\rangle \;.$$

Integrating out (tracing over/leaving out) the degrees of freedom of particle 2 results in a mixed ensemble for the QM description of particle 1. For instance this applies to scenarios in which only particle 1 can be measured, whereas particle 2 is virtually undetectable (as is for example the case for neutrinos).