Exercises for Quantum Mechanics 3
Set 8 (module 1)

This time you can choose from two sets of two exercises each.

- Exercises 15 and 16 correspond to the nuclear physics example that is treated in § 2.5.4 of the lecture notes.
- Exercises 17 and 18 correspond to the astrophysics example that is treated in § 2.5.5 of the lecture notes.

While preparing for the exam you can use the other set as practice material.

Exercise 15: Finite-size effects for Fermi gases (*relevant for heavy nuclei*)

Consider a spin-1/2 particle with mass m that is contained in a large cubic enclosure with impenetrable walls and fixed volume $V = L^3$. The influence from the edge of the cube is visible in the 1-particle energy spectrum and the spatial properties of the 1-particle energy eigenfunctions. Use the expressions for the energy eigenvalues and eigenfunctions from the lecture notes to answer the following questions.

Energy spectrum: the presence of the edge of the cube results in the quantum numbers $\nu_{x,y,z}$ being allowed to take positive integer values only. If V becomes very large, then the corresponding energy spectrum is approximately continuous. However, the energy eigenvalues for $\nu_x = 0 \vee \nu_y = 0 \vee \nu_z = 0$ should still be excluded.

(i) Argue that the number of quantum states with the length of the wave vector smaller than $k = \pi|\vec{\nu}|/L \equiv \pi\nu/L$ should be corrected as follows:

$$N(k) \approx \frac{1}{3}\pi\nu^3 - \frac{3}{4}\pi\nu^2 = \frac{V}{3\pi^2}\Big(k^3 - \frac{3\pi S}{8V}k^2\Big) \,,$$

with $S = 6L^2$ the total surface area of the sides of the cube.

Hint: use the partition of $\vec{\nu}$-space that was introduced during the lecture and subsequently argue what part of the volume should be left out.

(ii) Deduce from this the corrected density of states

$$D(E_{\rm kin}) \approx \frac{1}{2\pi^2}\Big(\frac{2m}{\hbar^2}\Big)^{3/2} V E_{\rm kin}^{1/2}\Big(1 - \frac{\pi S}{4V}\frac{\hbar}{\sqrt{2mE_{\rm kin}}}\Big) \,.$$

This expression actually applies to arbitrary shapes of the enclosure.

(iii) Presentation assignment: if you are not on the presentation team, you can treat this assignment as a bonus exercise.

- What does this surface correction imply for the ground-state energy of a Fermi gas of such spin-1/2 particles as compared to a situation without surface effects?
- What is in this context special about a spherical enclosure?
- What dou you expect to happen with the ground-state energy of the Fermi gas if the enclosure is split into two equal parts by means of an impenetrable wall?

Energy eigenfunctions: the impenetrable edge of the cube also translates into the constraint that the energy eigenfunctions should vanish on the edge. This means that the absolute value of the energy eigenfunctions can only become maximal at a certain distance from the edge.

(iv) Consider the edge at $x = 0$ and an energy eigenfunction with quantum number $\nu_x = n_F$. At what distance from the edge does the absolute value of this energy eigenfunction become maximal for the first time?

<u>Exercise 16</u>: Aspects of the Fermi-gas model for heavy nuclei

Consider for $T = 0$ a system that is comprised of two independent Fermi gases of protons and neutrons (with $M_p \approx M_n$), bound to a sphere with radius R by a constant potential $V = -V_0 < 0$ inside the sphere. Assume the system to consist of Z protons and $A-Z$ neutrons. The total (uniform) particle density of the system is given by

$$\rho_N \;=\; \frac{A}{4\pi R^3/3} \;=\; \frac{3}{4\pi r_0^3} \;=\; \text{constant} \; .$$

In terms of the ratio $2Z/A \equiv x$, this results in the proton and neutron densities

$$\rho_N^{(p)} \;=\; \frac{Z}{A}\,\rho_N \;=\; x\,\frac{\rho_N}{2} \qquad \text{and} \qquad \rho_N^{(n)} \;=\; \frac{A-Z}{A}\,\rho_N \;=\; (2-x)\,\frac{\rho_N}{2} \; .$$

For equal numbers of protons and neutrons the Fermi energies of the proton and neutron gases both amount to the same value $E_F \;=\; \left(\hbar^2/2M_p\right)\left(3\pi^2\rho_N/2\right)^{2/3}$.

(i) Show that for fixed A the lowest total kinetic energy is reached for $x = 1$, corresponding to a system that contains as many protons as neutrons. For convenience you are allowed to assume that Z can take all real values on the interval $[0, A]$.

(ii) Suppose that $x = 1 - \lambda$ with $0 \le \lambda \ll 1$, which implies that there is a slight excess of neutrons over protons. Perform a Taylor expansion around $\lambda = 0$ to demonstrate that the total kinetic energy of the system is approximately given by

$$E_{\text{tot, kin}}^{T=0} \;\approx\; \frac{3}{5}\,A E_F \,+\, \frac{\lambda^2 A}{3}\,E_F \;=\; \frac{3}{5}\,A E_F \,+\, \frac{(A-2Z)^2}{3A}\,E_F \; .$$

(iii) <u>Presentation assignment (cont'd)</u>: consider the effect on the binding potential caused by the uniform charge distribution induced by the protons. In analogy to the example in §2.5.5, the (classical) Coulomb energy of the spherical nucleus amounts to

$$\Delta V_{\text{Coul}} \;=\; \frac{3}{5}\,\frac{Z^2\alpha\hbar c}{R} \;=\; \frac{3\alpha\hbar c}{5r_0}\,\frac{Z^2}{A^{1/3}} \;=\; \frac{3\alpha\hbar c}{20r_0}\,A^{5/3}x^2 \; ,$$

given a total charge $+Z|e|$ inside the nucleus.

– Prove that for fixed A the total energy of the system is minimal if

$$x^{2/3} \,-\, (2-x)^{2/3} \,+\, \frac{3\alpha\hbar c}{5r_0 E_F}\,A^{2/3}\,x \;=\; 0 \; .$$

– Discuss the implications for the fraction of protons in heavy nuclei.

(iv) Suppose we want to perform a proper quantum mechanical calculation of the Coulomb corrections for proton pairs. Argue whether these corrections become larger or smaller than the classical result of part (iii) for

– the subset of interacting proton pairs that are in spin-triplet states;

– the subset of interacting proton pairs that are in spin-singlet states.

Exercise 17: **Ultra-relativistic gas of spin-1/2 particles at $T = 0$**

Consider an ultra-relativistic Fermi gas at $T = 0$. The gas consists of a vary large, constant number N of free spin-1/2 particles that are contained inside a macroscopic cubic enclosure with edges $L = V^{1/3}$ and impenetrable walls. In the ultra-relativistic limit the corresponding (kinetic) 1-particle energy eigenvalues are given by

$$E_\nu \;=\; \frac{\hbar\pi c}{L}\sqrt{\nu_x^2+\nu_y^2+\nu_z^2} \;\equiv\; \frac{\hbar\pi c}{L}\,\nu \qquad (\nu_{x,y,z} = 1,2,\cdots)\ ,$$

with c the speed of light in vacuum. Just as in the non-relativistic case, it will prove handy to use the quantized wave vector $\vec{k} = \pi\vec{\nu}/L$.

Use the expression for $D(k)$ given in the lecture notes to derive the corresponding ultra-relativistic expressions for the 1-particle density of states $D(E_{\rm kin})$, Fermi energy E_F, average kinetic energy per particle $\bar{E}_{\rm kin}$ and pressure P:

$$D(E_{\rm kin}) \;\approx\; \frac{VE_{\rm kin}^2}{\pi^2\hbar^3c^3} \quad , \quad E_F \;=\; \hbar c\,(3\pi^2\rho_N)^{1/3} \quad , \quad \bar{E}_{\rm kin} \;=\; \frac{3}{4}\,E_F \quad , \quad P \;=\; \frac{1}{4}\,\rho_N E_F\ ,$$

where ρ_N is the constant particle density of the Fermi gas.

Exercise 18: **Chandrasekhar limit for massive white dwarfs**

$\boxed{\text{Model 1}}$: the interior of a massive white dwarf is composed of one characteristic type of atom with charge number Z and mass number A. In first approximation all electrons have been pressure ionized to form an ultra-relativistic electron gas inside a spherical volume. Use the results of exercise 17 to answer the following questions.

(i) Show that the total energy of such a massive white dwarf can be approximated by

$$E_T \;=\; b\,\frac{M^{4/3}}{R} - \frac{3G_N M^2}{5R}\ , \qquad \text{with} \quad b \;=\; \frac{3}{4}\Big(\frac{9\pi}{4}\Big)^{1/3}\hbar c\,\Big(\frac{Z}{AM_p}\Big)^{4/3}\ .$$

Here M and R are the mass and radius of the white dwarf, G_N is Newton's gravitational constant, and $M_p \approx M_n$ is the proton mass.

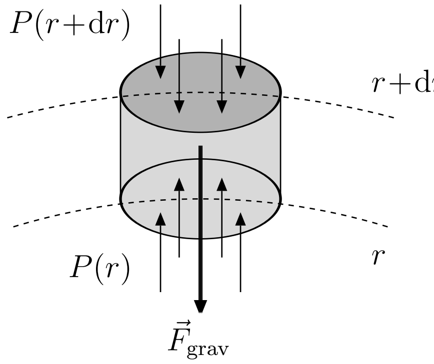
(ii) Presentation assignment: if you are not on the presentation team, you can treat this assignment as a bonus exercise.

– Why does the white dwarf continue to implode if $E_T < 0$?

– Argue that the white dwarf will eventually become stable if

$$E_T \;\geq\; 0 \quad\Rightarrow\quad M \;\leq\; \Big(\frac{5b}{3G_N}\Big)^{3/2} \overset{A=2Z}{=\!=} 3.423\times 10^{30}\,\text{kg} \;\approx\; 1.72 \ \text{solar masses}\ .$$

Used: $M_p^{-2}\big(\hbar c/G_N\big)^{3/2} \;=\; \dfrac{M_{\rm Planck}^3}{M_p^2} \;=\; 3.6851\times 10^{30}\,\text{kg} \;\approx\; 1.8533\ \text{solar masses}\ .$



$\boxed{\text{Model 2}}$: refine the previous model by replacing the constant mass density ρ_M of the spherical star by a more realistic radially symmetric density profile $\rho_M(r) \equiv \rho_c\theta^3(r)$, with $\theta(0) \equiv 1$ and with the radius R of the star defined through $\theta(R) \equiv 0$. Subsequently we assume that everywhere inside the star the gravitational force is balanced by the force induced by the electron gas pressure

$$P(r) \overset{\text{ex. 17}}{=\!=\!=} K\rho_M^{4/3}(r) , \qquad \text{with} \quad K = \frac{1}{4}\hbar c\,(3\pi^2)^{1/3}\Big(\frac{Z}{AM_p}\Big)^{4/3} .$$

To this end we consider an infinitesimal cylindrical volume element of the star with the axis of the cylinder pointing towards the center (see sketch). The two circular caps of the volume element each cover a surface area $\mathrm{d}A$ and are located at a distance r and $r+\mathrm{d}r$ from the center of the star.

(iii) Apply the force-equilibrium condition between gravity and gas pressure to derive that

$$m(r) = -\frac{r^2}{G_N\,\rho_M(r)}\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{4K}{G_N}\rho_c^{1/3}r^2\frac{\mathrm{d}\theta(r)}{\mathrm{d}r} ,$$

with $m(r)$ being the mass of the star inside a radial distance r from the center.

(iv) Use this to rewrite the definition

$$\rho_M(r) \equiv \frac{1}{4\pi r^2}\frac{\mathrm{d}m(r)}{\mathrm{d}r}$$

in terms of the dimensionless Lane–Emden equation

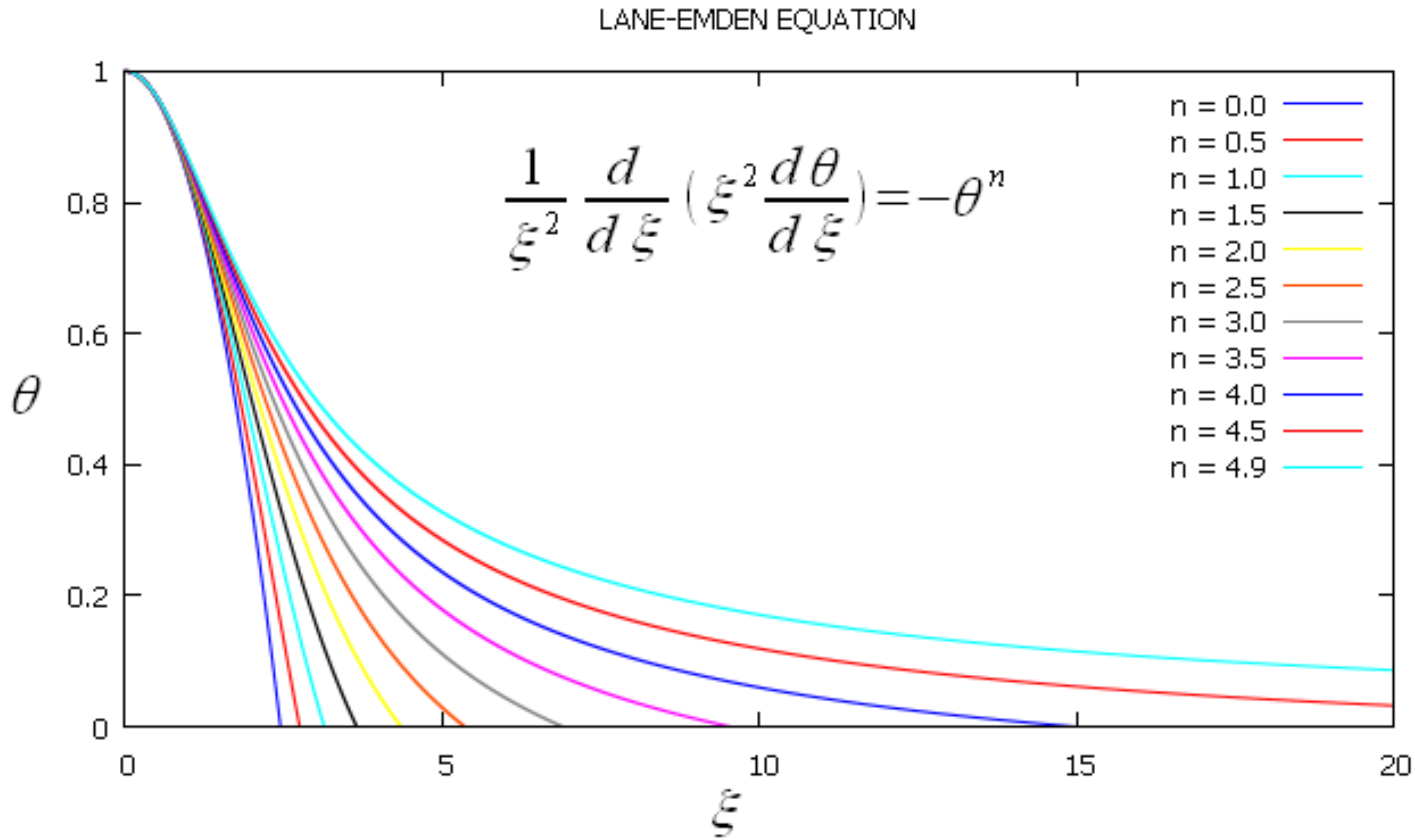
$$\theta^3 + \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\Big) = 0 , \qquad \text{with} \quad \xi \equiv r\rho_c^{1/3}\Big(\frac{\pi G_N}{K}\Big)^{1/2}.$$

This equation can be solved independent of the precise specifications of the star!

(v) Presentation assignment (cont'd): the desired solution to the Lane–Emden equation is fixed by the boundary conditions for $\xi = 0$, where $\theta = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi = 0$. The radius R of the star corresponds to $\xi = 6.897$, where $\theta = 0$ and $\xi^2\,\mathrm{d}\theta/\mathrm{d}\xi = -2.018$. Make use of part (iii) to demonstrate that this time there is a unique solution for the mass of the white dwarf, the so-called Chandrasekhar limit

$$M_C = m(R) = 8.072\,\pi\Big(\frac{K}{\pi G_N}\Big)^{3/2} \overset{A=2Z}{=\!=\!=} 2.854\times 10^{30}\,\mathrm{kg} \approx 1.44 \text{ solar masses} .$$

To be used: $M_p^{-2}\big(\hbar c/G_N\big)^{3/2} = \dfrac{M_{\mathrm{Planck}}^3}{M_p^2} = 3.6851\times 10^{30}\,\mathrm{kg} \approx 1.8533$ solar masses .



LANE-EMDEN EQUATION
$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})=-\theta^n$
n = 0.0
n = 0.5
n = 1.0
n = 1.5
n = 2.0
n = 2.5
n = 3.0
n = 3.5
n = 4.0
n = 4.5
n = 4.9
θ
ξ
0
0.2
0.4
0.6
0.8
1
5
10
15
20