Quantum Field Theory Exercises week 9

Exercise 12: loop corrections, on-shell internal particles and the optical theorem

Consider the ϕ^4-theory with $\underline{m=0}$ and look at the scattering reaction $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$.

(a) Calculate $\frac{1}{2}\int \mathrm{d}\Pi_2\, |\mathcal{M}(k_A, k_B \to p_1, p_2)|^2$ to lowest order in λ, with $\mathcal{M}(k_A, k_B \to p_1, p_2)$ the matrix element of the reaction and $\mathrm{d}\Pi_2$ the Lorentz invariant 2-body phase-space element. The factor 1/2 takes into account the fact that we are dealing with identical particles.

(b) Calculate $\mathcal{M}(k_A, k_B \to p_1, p_2)$ at one-loop order using a cut-off Λ^2 in the ℓ_E^2 integral, i.e.

$$\int_0^{\infty} \mathrm{d}\ell_E^2 \quad \xrightarrow{\text{to be replaced by}} \quad \int_0^{\Lambda^2} \mathrm{d}\ell_E^2 \ .$$

In the calculation you can use that Λ^2 is much larger than the Mandelstam variable s of the scattering reaction. The matrix element will consist of three Feynman diagrams that have a similar form: draw these diagrams. Once you have calculated one of them, you can find the others by interchanging the appropriate momenta. Note however that the three diagrams have a different analytic structure. As a result, while taking proper account of the sign of the relevant Mandelstam variable, you should keep a close eye on possible imaginary parts!

Hint: apply the two standard tricks for performing loop corrections discussed on pages 70-72 of the lecture notes.

Your intermediate result should effectively read:

$$\mathcal{M}(k_A, k_B \to p_1, p_2)_{1-loop} \;=\; \frac{\lambda^2}{32\pi^2}\left[\log\Big(\frac{\Lambda^2}{-s-i\epsilon}\Big) + \log\Big(\frac{\Lambda^2}{-t-i\epsilon}\Big) + \log\Big(\frac{\Lambda^2}{-u-i\epsilon}\Big) + 3\right] .$$

(c) Read § 2.9.4 of the lecture notes entitled "The optical theorem". Translated to exercise 12 the <u>optical theorem</u> states that $2\,\mathrm{Im}\big[\mathcal{M}(k_A, k_B \to p_1, p_2)\big]$ at one-loop order is equal to the quantity you calculated in (a) if there is no angular dependence. Check this equality explicitly.

(d) Now we want to see the explicit link with putting internal particles on-shell, as mentioned on page 80 of the lecture notes. To this end we consider the one-loop diagram that gave rise to the imaginary part of the matrix element in (c). Subsequently we put the two particles occurring inside the loop on their respective mass-shells by replacing both propagators by delta-functions according to $i/(q^2+i\epsilon) \to 2\pi\delta(q^2)$. Recalculate the loop correction after this replacement. Keep in mind that the integral is Lorentz invariant, so you can choose the CM frame of the reaction for expressing the external momenta.

Hint: this time you <u>don't</u> have to apply the standard tricks for performing loop corrections.

(e) Compare the results that you obtained in parts (c) and (d).