Extra exercise explaining minimal substitution

Classical electromagnetic fields and minimal substitution

Consider a spin-0 particle with mass m and charge q in a potential $V(\vec{r})$. The classical Lagrangian of this system is given by

$$L_0(\vec{r},\dot{\vec{r}} = \mathrm{d}\vec{r}/\mathrm{d}t) \;=\; \frac{1}{2}m\dot{\vec{r}}^{\,2} \;-\; V(\vec{r})\ ,$$

with corresponding Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial\dot{\vec{r}}} - \frac{\partial L_0}{\partial\vec{r}} \;=\; 0 \quad\Rightarrow\quad m\ddot{\vec{r}} \;=\; m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \;=\; -\vec{\nabla}V(\vec{r})\ .$$

In order to switch to the Hamiltonian H_0, we define the conjugate momentum $\vec{p}$ as

$$\vec{p} \;\equiv\; \frac{\partial L_0}{\partial\dot{\vec{r}}} \;=\; m\dot{\vec{r}}\ ,$$

so that

$$H_0(\vec{r},\vec{p}) \;\equiv\; \vec{p}\cdot\dot{\vec{r}} \;-\; L_0(\vec{r},\dot{\vec{r}}) \;=\; \frac{\vec{p}^{\,2}}{2m} \;+\; V(\vec{r})\ .$$

Subsequently we subject this system to a (classical) electromagnetic field, characterized by the real scalar potential $\phi(\vec{r},t)$ and the real vector potential $\vec{A}(\vec{r},t)$. This changes the classical Lagrangian into

$$L(\vec{r},\dot{\vec{r}},t) \;=\; L_0(\vec{r},\dot{\vec{r}}) \;+\; q\vec{A}(\vec{r},t)\cdot\dot{\vec{r}} \;-\; q\phi(\vec{r},t)\ .$$

(i) How does this affect the conjugate momentum $\vec{p} \equiv \partial L/\partial\dot{\vec{r}}$?

(ii) Show that the new Lagrange equation is given by

$$m\ddot{\vec{r}} \;=\; q\Big(\vec{\mathcal{E}}(\vec{r},t) \;+\; \dot{\vec{r}}\times\vec{\mathcal{B}}(\vec{r},t)\Big) \;-\; \vec{\nabla}V(\vec{r})\ ,$$

where the term proportional to q is the well-known <u>Lorentz force</u>.

Hint: use that $\vec{\nabla}(\vec{A}\cdot\dot{\vec{r}}) - (\dot{\vec{r}}\cdot\vec{\nabla})\vec{A} \;=\; \dot{\vec{r}}\times(\vec{\nabla}\times\vec{A})$, together with the definitions

$$\vec{\mathcal{E}}(\vec{r},t) \;\equiv\; -\vec{\nabla}\phi(\vec{r},t) \;-\; \frac{\partial}{\partial t}\vec{A}(\vec{r},t) \qquad\text{and}\qquad \vec{\mathcal{B}}(\vec{r},t) \;\equiv\; \vec{\nabla}\times\vec{A}(\vec{r},t)$$

for the electric and magnetic fields.

(iii) Demonstrate that the Hamiltonian is obtained by <u>minimal substitution</u>:

$$H(\vec{r},\vec{p},t) \;=\; \frac{\big[\vec{p} - q\vec{A}(\vec{r},t)\big]^2}{2m} \;+\; q\phi(\vec{r},t) \;+\; V(\vec{r})\ .$$

Minimal substitution provides a very direct and efficient way to implement the interaction between classical particles and classical electromagnetic fields within the Hamiltonian formalism.