As in the case of the Yukawa interactions, also the local electromagnetic interactions between the matter particles are mediated by force carriers. This was to be expected, bearing in mind that charged objects are observed to interact while being at non-zero disctance! Since $[\psi] = [\bar{\psi}] = 3/2$ and $[A_\mu] = 1$, the electric charge $q \in \mathbb{R}$ is a dimensionless coupling constant. We will see later that this dimensionless coupling constant indeed implies that QED is a renormalizable theory.

QED from a symmetry principle: local gauge invariance (gauge principle).

Alternatively we could start from the free Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}}(x) \;=\; i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \;-\; m\bar{\psi}(x)\psi(x) \;,$$

which is invariant under the global gauge transformation (abelian $U(1)$ transformation)

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \quad , \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha}\bar{\psi}(x) \qquad (\alpha \in \mathbb{R} \text{ independent of } x^\mu) \;.$$

According to Noether's theorem this global gauge symmetry can be associated with a conserved current and charge. In non-relativistic quantum mechanics this global gauge invariance of a free-fermion system simply underlines the unobservability of the absolute phase of a wave function: only relative phases are observable through interference.

13c *The gauge principle: in the context of relativistic gauge theories, which should be local, it is now postulated that this gauge invariance should also hold locally.*

Consider to this end the local gauge transformation

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x) \quad , \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha(x)}\bar{\psi}(x) \qquad (\alpha(x) \text{ a real scalar field}) \;.$$

The requirement of local gauge invariance[5] has profound consequences, since the kinetic term transforms as

$$i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \;\to\; i\bar{\psi}(x)e^{-i\alpha(x)}\gamma^\mu\partial_\mu\big[e^{i\alpha(x)}\psi(x)\big] \;=\; i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \bar{\psi}(x)\gamma^\mu\psi(x)\big[\partial_\mu\alpha(x)\big]$$

and therefore is not invariant under local gauge transformations. The last term, which involves the covariant vector field $\partial_\mu\alpha(x)$, explicitly spoils the invariance. So, we need to replace the ordinary derivative ∂_μ by a gauge covariant derivative (or short: covariant derivative) D_μ such that

$$D_\mu\psi(x) \;\to\; D'_\mu\psi'(x) \;=\; e^{i\alpha(x)}D_\mu\psi(x) \;,$$

causing $D_\mu\psi(x)$ and $\psi(x)$ to transform similarly under local gauge transformations! This can be achieved by

$$D_\mu \;\equiv\; \partial_\mu + igA_\mu(x) \;, \quad \text{with} \quad A_\mu(x) \;\to\; A'_\mu(x) \;=\; A_\mu(x) - \frac{1}{g}\,\partial_\mu\alpha(x) \;,$$

[5] See the bachelor thesis of Pim van Oirschot for more details and extra motivation

where g is a gauge coupling and $A_\mu(x)$ a gauge field. In view of the Lorentz transformation property of $\partial_\mu\alpha(x)$, this gauge field should be a covariant vector field. Its transformation property resembles a gauge transformation for the electromagnetic vector potential with $\chi(x) = -\alpha(x)/\mathrm{g}$. This observed gauge-freedom redundancy in the electromagnetic description is exploited here to reveal the more profound local gauge invariance of QED!

Proof:

$$D'_\mu\psi'(x) = \Big(\partial_\mu + i\mathrm{g}\big[A_\mu(x) - \frac{1}{\mathrm{g}}\partial_\mu\alpha(x)\big]\Big)e^{i\alpha(x)}\psi(x)$$

$$= e^{i\alpha(x)}\big[\partial_\mu\psi(x) + i\psi(x)\partial_\mu\alpha(x) + i\mathrm{g}A_\mu(x)\psi(x) - i\psi(x)\partial_\mu\alpha(x)\big] = e^{i\alpha(x)}D_\mu\psi(x)\,.$$

This means that the Lagrangian

$$i\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) - m\bar{\psi}(x)\psi(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) - \mathrm{g}\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$$

is locally gauge invariant. It contains the gauge interaction

$$\mathcal{L}_{\text{int}}(x) = -\mathrm{g}\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\,,$$

which involves a gauge field that is coupled to a conserved current. Finally we can add the gauge-invariant kinetic term $\mathcal{L}_{\text{Maxwell}}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$ for a free gauge field, where the field tensor $F_{\mu\nu}(x)$ is defined as

$$i\mathrm{g}F_{\mu\nu}(x) \equiv \big[D_\mu, D_\nu\big] = \big[\partial_\mu + i\mathrm{g}A_\mu(x)\big]\big[\partial_\nu + i\mathrm{g}A_\nu(x)\big] - \big[\partial_\nu + i\mathrm{g}A_\nu(x)\big]\big[\partial_\mu + i\mathrm{g}A_\mu(x)\big]$$

$$= i\mathrm{g}\big[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\big]\,.$$

In conclusion, for $\mathrm{g} = |e|$ we find the same Lagrangian $\mathcal{L}_{\text{QED}}$ as obtained by minimal substitution for a particle with charge $+|e|$. For a general charge $q = Q|e|$ one has to modify the gauge transformation according to $e^{i\alpha(x)} \to e^{iQ\alpha(x)}$ and the covariant derivative according to $D_\mu \to \partial_\mu + iQ|e|A_\mu(x) = \partial_\mu + iqA_\mu(x)$. Such a rescaling leaves the transformation property of the gauge field unaffected, but changes the interaction strength from $|e|$ to q.

Massless gauge fields: *a massive gauge field would correspond to an extra mass term* $+\frac{1}{2}M_A^2\,A_\mu(x)A^\mu(x)$ *in the Lagrangian, which is obviously not gauge invariant. A theory that is manifestly invariant under local gauge transformations requires the gauge bosons described by* $A_\mu(x)$ *to be massless, i.e.* $M_A = 0$. *So, in order to give mass to gauge bosons an additional mechanism is required in the context of gauge theories.*

13c *Going beyond QED: motivated by the success of describing QED through the gauge principle, this postulate will later on be extended to other types of* *gauge transformations in order to describe other fundamental interactions in nature, i.e. the strong and weak interactions. The associated extended gauge interactions will describe the fundamental interactions between matter fermions as being mediated by gauge bosons, just like we have just worked out for the electromagnetic interactions that are mediated by photons. In order to find the right group structure for the extended gauge transformations, we will be guided by experimental observations of particle interactions and charge conservation laws!*

5.2.1 Quantization of the free electromagnetic theory

The gauge freedom of the electromagnetic vector potential complicates the usual quantization procedure. The reason for this lies in the following observations.

The electromagnetic gauge freedom revisited:

- The gauge freedom for non-constant $\chi(x)$ reflects the redundancy in our description of electromagnetism: the gauge-transformed fields describe the same physics and are therefore to be identified. This can be traced back to the electromagnetic wave equation

$$\Box A^{\nu}(x) - \partial^{\nu}\big(\partial_{\mu}A^{\mu}(x)\big) = (g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})A^{\mu}(x) = j_c^{\nu}(x) \ ,$$

where the differential operator $(g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})$ is not invertible in the Green's function sense as $(g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})\partial^{\mu}\chi(x) = 0$ for arbitrary $\chi(x)$. Given an initial field configuration $A^{\mu}(t_0,\vec{x}\,)$ we cannot unambiguously determine $A^{\mu}(t,\vec{x}\,)$, since $A^{\mu}(x)$ and $A^{\mu}(x) + \partial^{\mu}\chi(x)$ are not distinguishable.

13a *Hence, $A^{\mu}(x)$ is actually not a physical object as it contains redundant information! All fields that are linked by a gauge transformation form an equivalence class and are therefore to be identified: the physics is uniquely described by selecting a representative of each equivalence class. Different configurations of these representatives are called different gauges. By fixing the gauge the redundancy is removed and an unambiguous electromagnetic evolution is obtained. We can choose freely here, but some choices will prove more handy for certain problems than others.*

- By choosing an appropriate $\chi(x)$ it is possible to cast $A_{\mu}(x)$ in such a form that the Coulomb condition $\vec{\nabla}\cdot\vec{A}^{\rm trans}(x) = A_0^{\rm trans}(x) = 0$ is satisfied. In this form we see immediately that $A_{\mu}^{\rm trans}(x)$ has in fact only two physical (transverse) degrees of freedom! These are the degrees of freedom that should be quantized in the corresponding quantum field theory ... however, the Coulomb condition is not Lorentz invariant and therefore leads to Feynman rules that are rather unpleasant.

- Lorentz invariance is manifest, resulting in simple Feynman rules, if we choose $\chi(x)$ such that the Lorenz condition $\partial \cdot A(x) = 0$ is satisfied. In this form we do not see straightaway that $A^\mu(x)$ has two physical degrees of freedom. One would expect three physical degrees of freedom in view of the Lorenz condition $\partial \cdot A(x) = 0$, but there is still more gauge freedom left as a result of the gauge transformation $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \chi'(x)$ with $\Box \chi'(x) = 0$.

Quantized free electromagnetic field: the quantized electromagnetic theory should reproduce the classical Maxwell theory in the classical limit. Due to the correspondence principle this implies that the above-given gauge-fixing conditions are to be implemented as expectation values for physical (asymptotic) Fock states $|\psi\rangle$. As a direct consequence of implementing the Lorenz condition $\langle\psi|\partial \cdot \hat{A}(x)|\psi\rangle = 0$, all relevant components of the electromagnetic potential satisfy the massless KG equation $\Box A_\mu(x) = 0$. In the Coulomb gauge we can therefore quantize as in the massless scalar case:

$$\hat{A}_\mu^{\rm trans}(x) \overset{\hat{A}_\mu^\dagger(x) = \hat{A}_\mu(x)}{=\!=\!=\!=} \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1}^{2} \left(\hat{a}_{\vec{p}}^r \epsilon_\mu^r(p) e^{-ip\cdot x} + \hat{a}_{\vec{p}}^{r\dagger} \epsilon_\mu^{r*}(p) e^{ip\cdot x} \right)\Big|_{p_0 = E_{\vec{p}} = |\vec{p}|} ,$$

in terms of the two physical transverse polarization vectors

$$\epsilon_0^1(p) = \epsilon_0^2(p) = 0 \quad , \quad \vec{\epsilon}^{\,1}(p) \cdot \vec{p} = \vec{\epsilon}^{\,2}(p) \cdot \vec{p} = 0$$

with normalization condition $\epsilon^r(p) \cdot \epsilon^{r'*}(p) = -\,\delta_{rr'}$. The creation and annihilation operators $\hat{a}_{\vec{p}}^{r\dagger}$ and $\hat{a}_{\vec{p}}^r$ of the

13d *massless electromagnetic spin-1 energy quanta (photons = antiphotons)*

satisfy the bosonic quantization conditions

$$\left[\hat{a}_{\vec{p}}^r, \hat{a}_{\vec{p}'}^{r'\dagger}\right] = (2\pi)^3 \delta_{rr'} \delta(\vec{p} - \vec{p}\,')\hat{1} \qquad \text{and} \qquad \left[\hat{a}_{\vec{p}}^r, \hat{a}_{\vec{p}'}^{r'}\right] = \left[\hat{a}_{\vec{p}}^{r\dagger}, \hat{a}_{\vec{p}'}^{r'\dagger}\right] = 0 .$$

If we replace the Coulomb condition by the Lorenz condition, the two versions of the electromagnetic field are linked by the identity $\langle\psi|\hat{A}_\mu(x)|\psi\rangle = \langle\psi|\hat{A}_\mu^{\rm trans}(x)|\psi\rangle + \partial_\mu \chi(x)$, with $\Box \chi(x) = 0$. This identity reflects the remaining gauge arbitrariness of the classical electromagnetic field $\langle\psi|\hat{A}_\mu(x)|\psi\rangle$ in the Lorenz gauge.

Feynman propagator and polarization sum: for performing Feynman-diagram calculations we need one more ingredient, the photon propagator. The amplitude for the propagation of photons from y to x reads

$$\langle 0|\hat{A}_\mu^{\rm trans}(x)\hat{A}_\nu^{\rm trans}(y)|0\rangle = \int \frac{d\vec{p}\,d\vec{p}\,'}{(2\pi)^6} \frac{e^{-ip\cdot x + ip'\cdot y}}{2\sqrt{E_{\vec{p}} E_{\vec{p}'}}} \sum_{r,r'=1}^{2} \epsilon_\mu^r(p) \epsilon_\nu^{r'*}(p') \langle 0|\hat{a}_{\vec{p}}^r \hat{a}_{\vec{p}'}^{r'\dagger}|0\rangle \Big|_{p_0 = |\vec{p}|,\, p'_0 = |\vec{p}\,'|}$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \frac{e^{-ip\cdot(x-y)}}{2E_{\vec{p}}} \sum_{r=1}^{2} \epsilon_\mu^r(p) \epsilon_\nu^{r*}(p) \Big|_{p_0 = |\vec{p}|} .$$

This expression for the propagation amplitude is rather awkward, since it involves the so-called polarization sum for external (physical) photons:

$$\sum_{r=1}^{2} \epsilon^r_\mu(p)\epsilon^{r\,*}_\nu(p) \;=\; -\,g_{\mu\nu} \;-\; \frac{p_\mu p_\nu}{(n\cdot p)^2} \;+\; \frac{p_\mu n_\nu + n_\mu p_\nu}{n\cdot p}\;,$$

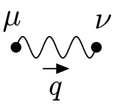
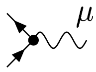
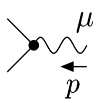
expressed in terms of the temporal unit vector $n_\mu \equiv (1,\vec{0}\,)$. Such a complicated expression is unavoidable for external photons and for the propagator in the Coulomb gauge, but we can exploit the gauge freedom in the Lorenz gauge to remove all terms $\propto\, p_\mu\,, p_\nu$ (see § 5.5). In this so-called 't Hooft-Feynman gauge the photon propagator reduces to

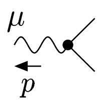
$$\langle 0|T\big(\hat{A}_\mu(x)\hat{A}_\nu(y)\big)|0\rangle \;=\; \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{-\,ig_{\mu\nu}}{p^2+i\epsilon}\;e^{-ip\cdot(x-y)} \;=\; -\,g_{\mu\nu}\,D_F(x-y;m^2=0)\;.$$

14a *The propagator for internal (virtual) photons has become extremely simple and manifestly Lorentz covariant in the 't Hooft-Feynman gauge!*

5.3 Additional QED Feynman rules in 't Hooft-Feynman gauge

In order to obtain the full set of momentum-space Feynman rules for QED we simply have to supplement the Feynman rules for fermions, which were given in the context of the Yukawa theory, by the following four photonic Feynman rules:

1. For each photon propagator  insert $\dfrac{-\,ig_{\mu\nu}}{q^2+i\epsilon}$.
2. For each QED vertex  insert $-\,iq\gamma^\mu$.
3. For each incoming photon line  $=\; \hat{A}_\mu(x)|\vec{p},r\rangle_0$ insert $\epsilon^r_\mu(p)\sqrt{Z_3}$ $\quad(r=1,2)$.

   For each outgoing photon line  $=\; {}_0\langle\vec{p},r|\hat{A}_\mu(x)$ insert $\epsilon^{r\,*}_\mu(p)\sqrt{Z_3}$ $\quad(r=1,2)$.

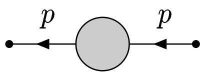
The following remarks are in order. First of all, the polarization vectors featuring in the last two Feynman rules are transverse (physical) ones and $\sqrt{Z_3}$ is the wave-function renormalization factor for photons. Secondly, the sign (direction) of the momentum in the photon propagator does not matter, like in the scalar case. Finally, the γ-matrix occurring in the QED vertex is a 4×4 matrix in spinor space that will be contracted with other 4×4 matrices and/or spinors, with the Dirac indices contracted as usual along the fermion line against the arrow.

14a *Remark: since $\langle\psi|\hat{A}_\mu(x)|\psi\rangle = \langle\psi|\hat{A}^{trans}_\mu(x)|\psi\rangle + \partial_\mu\chi(x)$ with $\Box\chi(x) = 0$, we can always add to $\epsilon^r_\mu(p)$ a term $\propto p_\mu$ with $p^2 = 0$ without changing the physics outcome (see § 5.5).*

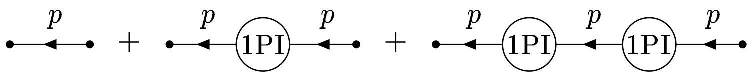
5.4 Full fermion propagator (§ 7.1 in the book)

To all orders in perturbation theory the full fermion propagator in QED is given by the Dyson series

$$\int \mathrm{d}^4x \; e^{ip\cdot x} \langle \Omega | T\big(\hat{\psi}(x)\hat{\bar{\psi}}(0)\big)|\Omega\rangle \quad \equiv$$



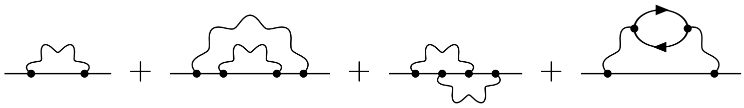
$$=$$



$$+ \; \cdots \; ,$$

where

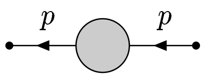
$$\equiv \; -i\Sigma(\not{p}) \; =$$



$$+ \; \cdots$$

is the collection of all 1-particle irreducible fermion self-energy diagrams. This Dyson series can again be summed up as a geometric series:

$$\int \mathrm{d}^4x \; e^{ip\cdot x} \langle \Omega | T\big(\hat{\psi}(x)\hat{\bar{\psi}}(0)\big)|\Omega\rangle \quad =$$



$$= \quad \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} + \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\Big(-i\Sigma(\not{p})\Big)\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} + \cdots$$

$$= \quad \frac{i}{\not{p}-m-\Sigma(\not{p})} \;\equiv\; S(p) \; ,$$

using that $\Sigma(\not{p}) = \Sigma_S(p^2)\,m + \Sigma_V(p^2)\,\not{p}$ commutes with $\not{p}$ and the mass parameter m in the Lagrangian. The full propagator has a simple pole located at the physical mass m_{ph}, which is shifted away from m by the fermion self-energy:

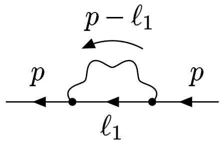
$$\Big[\not{p}-m-\Sigma(\not{p})\Big]\bigg|_{\not{p}=m_{ph}} = \; 0 \quad \Rightarrow \quad m_{ph}-m-\Sigma(\not{p}=m_{ph}) \; = \; 0 \; .$$

Close to this pole the denominator of the full propagator can be expanded according to

$$\not{p}-m-\Sigma(\not{p}) \;\approx\; (\not{p}-m_{ph})\big[1-\Sigma'(\not{p}=m_{ph})\big] \;+\; \mathcal{O}\big([\not{p}-m_{ph}]^2\big) \quad \text{for} \quad \not{p}\approx m_{ph} \; ,$$

where $\Sigma'(\not{p})$ stands for the derivative of the fermion self-energy with respect to $\not{p}$. Just like in the Källén–Lehmann spectral representation, the full propagator has a single-particle pole of the form $iZ_2(\not{p}+m_{ph})/(p^2-m_{ph}^2+i\epsilon)$ with $Z_2 = 1/\big[1-\Sigma'(\not{p}=m_{ph})\big]$ (see p. 115).

The fermion self-energy: in order to find out whether the fermion self-energy is more difficult to calculate we consider the 1-loop contribution in QED. Indicating the photon mass by λ we then obtain



$$= -i\Sigma_2(\not{p}) = (-iq)^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \gamma^\mu \frac{i(\not{\ell}_1 + m)}{\ell_1^2 - m^2 + i\epsilon}\, \gamma^\nu \frac{-ig_{\mu\nu}}{(p-\ell_1)^2 - \lambda^2 + i\epsilon}\Bigg|_{\lambda\downarrow 0}$$

$$= -q^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \frac{4m - 2\not{\ell}_1}{\big[\ell_1^2 - m^2 + i\epsilon\big]\big[(p-\ell_1)^2 - \lambda^2 + i\epsilon\big]}$$

$$\underline{\underline{\text{p. } 70\,,\ \ell_1 = \ell + \alpha_2 p}} \quad -q^2 \int_0^1 \mathrm{d}\alpha_2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{4m - 2\not{\ell} - 2\alpha_2\not{p}}{(\ell^2 - \Delta + i\epsilon)^2}$$

$$= -q^2 \int_0^1 \mathrm{d}\alpha_2\, (4m - 2\alpha_2\not{p}) \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{1}{(\ell^2 - \Delta + i\epsilon)^2}\ ,$$

with

$$\Delta = \alpha_2\lambda^2 + (1-\alpha_2)m^2 - \alpha_2(1-\alpha_2)p^2$$

just like in the scalar case. In the second line of this expression we have used that

$$\gamma^\mu(\not{\ell}_1 + m)\gamma_\mu = (m - \not{\ell}_1)\gamma^\mu\gamma_\mu + 2\not{\ell}_1 = 4m - 2\not{\ell}_1\ .$$

The threshold for the creation of a fermion–photon 2-particle state is here situated at $p^2 = (m+\lambda)^2$, which approaches m^2 in the limit $\lambda\downarrow 0$ for massless photons. The rest of the calculation, including the regularization of the UV divergence, goes like in the scalar case worked out in § 2.9.2. Note that the fermion mass receives a UV-divergent shift $\Sigma_2(\not{p} = m_{ph}) \propto m_{ph}\log(\Lambda^2/m_{ph}^2)$.

14b *Fermion masses are naturally protected against high-scale quantum corrections: if there would be no coupling between left- and right-handed Dirac fields in the Dirac Lagrangian (i.e. $m = 0$), then no such coupling can be induced by the perturbative vector-current QED corrections! Fermion masses are protected by the invariance under chiral transformations of the massless theory.*

5.5 The Ward–Takahashi identity in QED (§ 7.4 in the book)

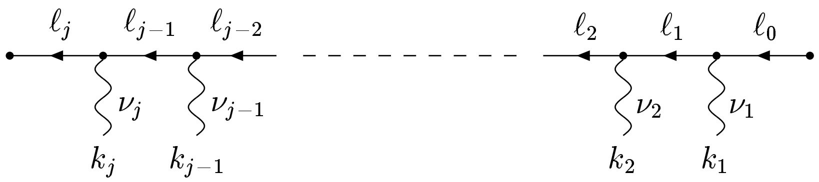
14c *Question: how does the gauge invariance of QED manifest itself in Green's functions and scattering amplitudes?*

In order to answer this question, we consider a QED diagram to which we want to attach an additional on/off-shell photon with momentum k. Upon contraction of this photon line with the corresponding momentum k a special identity can be derived that is related to the $U(1)$ gauge symmetry. After all, local $U(1)$ gauge invariance causes the photon field to couple to a conserved current resulting from charged matter. By replacing the photon field by its momentum we perform the effective momentum-space replacement $A_\mu \to \partial_\mu$, which should produce a vanishing result when applied to a full-fledged conserved current.

Step 1: how can the photon be attached to an arbitrary diagram involving (anti)fermions and photons?

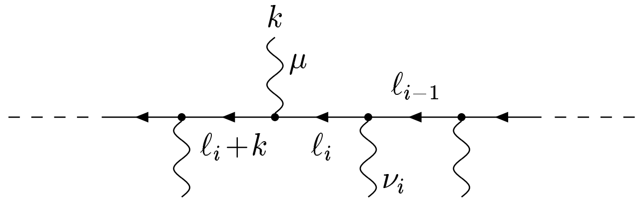
- The photon cannot be attached to a photon, since it has charge 0.
- The photon can be attached to a fermion line that connects two external points or to a fermion loop.

Step 2: consider an arbitrary fermion line with j photons attached to it and all photon momenta defined to be incoming. Graphically this can be represented by



where $\ell_i = \ell_0 + \sum_{n=1}^{i} k_n$. This line can either flow between external points or close into a loop (which means that $l_0 = l_j$) and the photons can either be on-shell or virtual. There will be $j+1$ places to insert the extra photon with momentum k, for example between photons i and $i+1$:

k_μ *



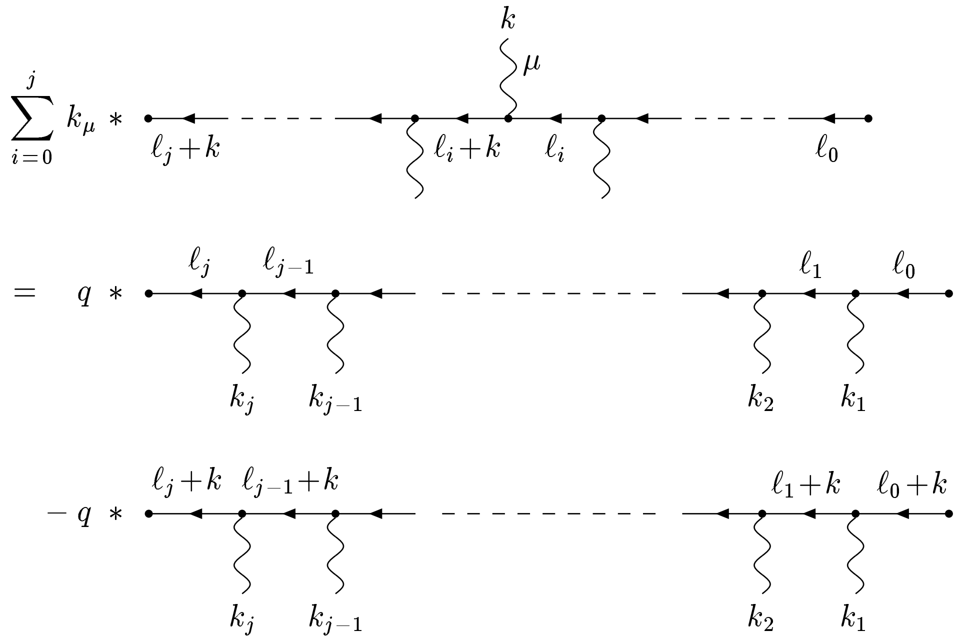
$$= \cdots \left[\frac{i}{\not{\ell}_i + \not{k} - m} (-iq\not{k}) \frac{i}{\not{\ell}_i - m} (-iq\gamma^{\nu_i}) \frac{i}{\not{\ell}_{i-1} - m} \right] \cdots$$

$$= \cdots \left[q \left(\frac{i}{\not{\ell}_i - m} - \frac{i}{\not{\ell}_i + \not{k} - m} \right) (-iq\gamma^{\nu_i}) \frac{i}{\not{\ell}_{i-1} - m} \right] \cdots ,$$

where we have used that $\not{k} = \not{\ell}_i + \not{k} - m - (\not{\ell}_i - m)$ in the last step. Insertion between photons $i-1$ and i gives in a similar way:

$$\cdots \left[\frac{i}{\not{\ell}_i + \not{k} - m} (-iq\gamma^{\nu_i}) \, q \left(\frac{i}{\not{\ell}_{i-1} - m} - \frac{i}{\not{\ell}_{i-1} + \not{k} - m} \right) \right] \cdots .$$

Note that the second term of the i^{th} insertion cancels the first term of the $(i-1)^{\text{th}}$ insertion. Finally we have to sum over all possible insertions along the fermion line. This causes all terms to cancel pairwise except for two unpaired terms at the very end of the chain:



As soon as all charge is accounted for, the fermion line represents a conserved current and the right-hand-side of the above identity should vanish! This happens in two distinct cases.

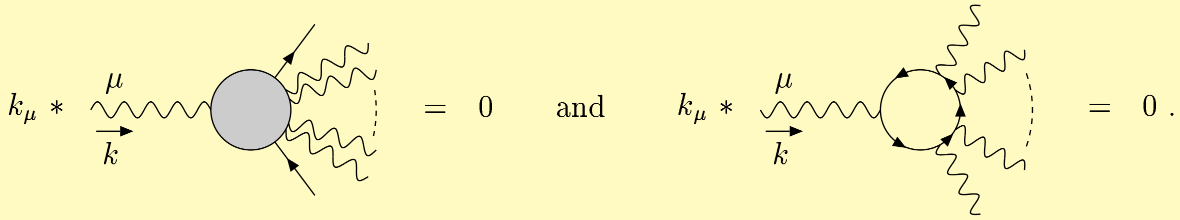
Case 1: if the fermion line is part of an on-shell matrix element and connects two of the external (asymptotic) fermion states, then the corresponding amputation procedure removes both terms on the right-hand-side. This is caused by the fact that one of the endpoints gives rise to a shifted 1-particle pole, i.e. $1/(\ell_j^2-m^2)$ instead of $1/[(\ell_j+k)^2-m^2]$ or $1/[(\ell_0+k)^2-m^2]$ instead of $1/(\ell_0^2-m^2)$.

Case 2: if the fermion line closes in itself to form a loop (i.e. $\ell_0 = \ell_j + k$), then the two terms on the right-hand-side give rise to the integrals

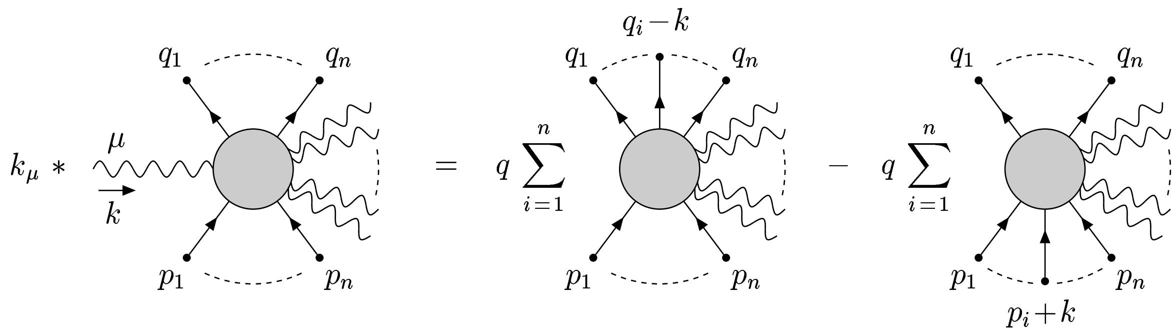
$$-\,q^{j+1}\int \frac{\mathrm{d}^4\ell_0}{(2\pi)^4}\left[\mathrm{Tr}\Big(\frac{1}{\not{\ell}_0-m}\,\gamma^{\nu_j}\,\frac{1}{\not{\ell}_{j-1}-m}\,\gamma^{\nu_{j-1}}\,\cdots\,\frac{1}{\not{\ell}_1-m}\,\gamma^{\nu_1}\Big)\right.$$
$$\left. -\,\mathrm{Tr}\Big(\frac{1}{\not{\ell}_0+\not{k}-m}\,\gamma^{\nu_j}\,\frac{1}{\not{\ell}_{j-1}+\not{k}-m}\,\gamma^{\nu_{j-1}}\,\cdots\,\frac{1}{\not{\ell}_1+\not{k}-m}\,\gamma^{\nu_1}\Big)\right] \;=\; 0\;,$$

if we are allowed to change the integration variable from ℓ_0 to ℓ_0+k in the first term!

Diagrammatically this can be summarized by the following two Ward – Takahashi identities for on-shell amplitudes and fermion loops:



More general identity: the Ward – Takahashi identity for Green's functions reads

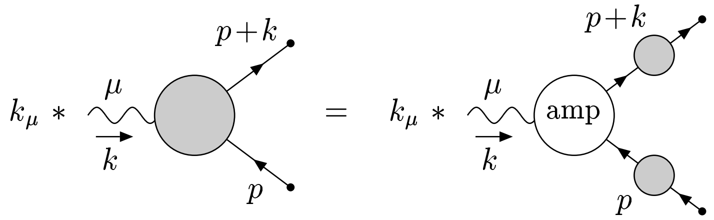


where the blobs represent all possible diagrams and photon insertions. In formula language this can be written compactly as

$$k_\mu G^\mu(k; p_1, \cdots, p_n; q_1, \cdots, q_n) \;=\; q \sum_i \Big(G(p_1, \cdots, p_n; q_1, \cdots, q_{i-1}, q_i - k, q_{i+1}, \cdots, q_n) \\ -\; G(p_1, \cdots, p_{i-1}, p_i + k, p_{i+1}, \cdots, p_n; q_1, \cdots, q_n) \Big)\;.$$

14c *This is the diagrammatic identity that imposes the $U(1)$ gauge symmetry and associated electric charge conservation on quantum mechanical amplitudes!*

Example of a Ward – Takahashi identity:



$$= \; S(p+k)\Big(-iq\, k_\mu \Gamma^\mu(p+k,p)\Big) S(p) \;\; \overset{\text{Ward – Takahashi}}{=\!=\!=\!=\!=} \;\; q\, S(p) - q\, S(p+k)$$

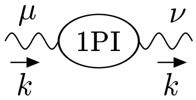
$$\Rightarrow \; -ik_\mu \Gamma^\mu(p+k,p) \;=\; S^{-1}(p+k) - S^{-1}(p) \;=\; -i\left[\not{k} + \Sigma(\not{p}) - \Sigma(\not{p}+\not{k})\right]\;.$$

Here $S(p)$ is the full fermion propagator, $\Sigma(\not{p})$ the corresponding 1-particle irreducible self-energy and $-iq\,\Gamma^\mu(p+k,p)$ the sum of all amputated 3-point diagrams contributing to the QED vertex. Hence, $\Gamma^\mu(p+k,p)$ is given by γ^μ at lowest order in perturbation theory, which is indeed in agreement with the Ward – Takahashi identity.

5.6 The photon propagator (§ 7.5 in the book)

The Ward – Takahashi identity has important implications for the properties of the photon propagator.

Transversality: the 1-particle irreducible photon self-energy

$$i\Pi^{\mu\nu}(k) \;\equiv$$ 

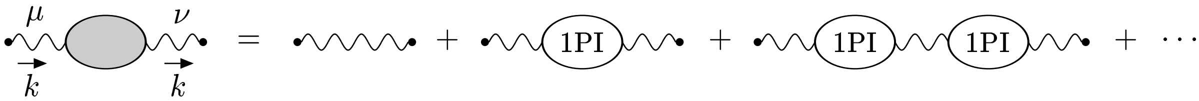
satisfies the Ward – Takahashi identity (transversality condition)

$$k_\mu \Pi^{\mu\nu}(k) \;=\; 0 \,.$$

In view of Lorentz covariance $\Pi^{\mu\nu}(k)$ can be decomposed into only two possible terms, a term $\propto g^{\mu\nu}$ and a term $\propto k^\mu k^\nu$. Therefore the Ward – Takahashi identity translates into the condition

$$\Pi^{\mu\nu}(k) \;=\; (k^2 g^{\mu\nu} - k^\mu k^\nu)\,\Pi(k^2) \,,$$

with $\Pi(k^2)$ regular at $k^2 = 0$ since a pole at $k^2 = 0$ would imply the existence of a single-massless-particle intermediate state. As a result, the full photon propagator is of the form

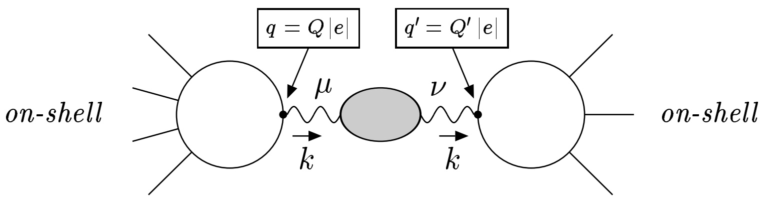


$$= \frac{-ig_{\mu\nu}}{k^2+i\epsilon} + \frac{-ig_{\mu\rho}}{k^2+i\epsilon}\Big[i(k^2 g^{\rho\sigma} - k^\rho k^\sigma)\,\Pi(k^2)\Big]\frac{-ig_{\sigma\nu}}{k^2+i\epsilon} + \cdots$$

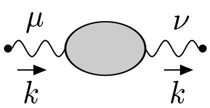
$$= -\frac{i}{k^2+i\epsilon}\,(g_{\mu\nu} - k_\mu k_\nu/k^2)\,\big[1 + \Pi(k^2) + \cdots\big] - \frac{i}{k^2+i\epsilon}\,\frac{k_\mu k_\nu}{k^2}$$

$$= \frac{-i(g_{\mu\nu} - k_\mu k_\nu/k^2)}{(k^2+i\epsilon)\big[1-\Pi(k^2)\big]} - \frac{i}{k^2+i\epsilon}\left(\frac{k_\mu k_\nu}{k^2}\right) .$$

Mass of the photon: consider an arbitrary internal photon line



The k_μ and k_ν terms in the full propagator yield a vanishing contribution due to the Ward – Takahashi identity for on-shell amplitudes. Hence,



$$\xrightarrow{\text{effectively}} \frac{-ig_{\mu\nu}}{(k^2+i\epsilon)\big[1-\Pi(k^2)\big]} \,,$$

which has a pole at $k^2 = 0$ with residue $Z_3 \equiv \big[1 - \Pi(0)\big]^{-1}$. As a result of the Ward–Takahashi identity, which in turn is a consequence of the gauge symmetry, $m_{\text{photon}} = 0$ to all orders in perturbation theory:

14b + 14c *the local $U(1)$ gauge symmetry protects the photon from becoming massive through quantum corrections.*

Observable charge: consider the same amplitude as before for

- low $|k^2| \;\Rightarrow\; e \to e\sqrt{Z_3}$, which is the finite physically observable charge obtained from the singular quantities e and Z_3;
- high $|k^2| \;\Rightarrow\; \dfrac{-ig_{\mu\nu}e^2}{k^2} \to \dfrac{-ig_{\mu\nu}e^2}{k^2\big[1-\Pi(k^2)\big]} = \dfrac{-ig_{\mu\nu}Z_3e^2}{k^2\Big(1 - Z_3\big[\Pi(k^2)-\Pi(0)\big]\Big)}$

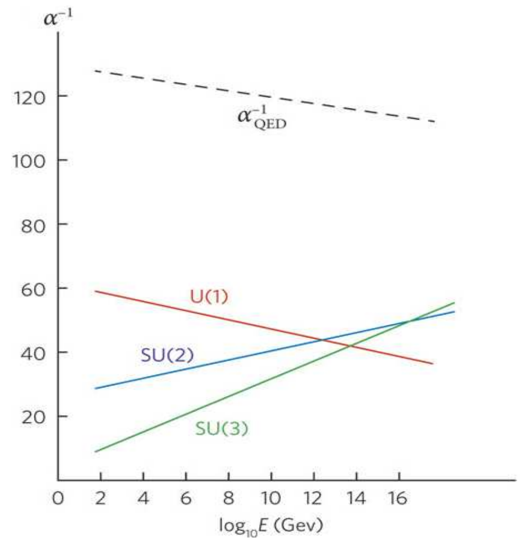
$$\Rightarrow \frac{e^2}{4\pi} = \alpha \to \alpha(k^2) = \frac{Z_3\,\alpha}{1 - Z_3\big[\Pi(k^2)-\Pi(0)\big]}\,,$$

where the factor Z_3 in front of $\big[\Pi(k^2)-\Pi(0)\big]$ turns e^2 inside the photon self-energy into the finite combination Z_3e^2.

14d *The electromagnetic fine structure constant becomes a running coupling, i.e. a coupling that changes with invariant mass. In fact it becomes larger with increasing invariant mass, causing the exchanged (virtual) photon to propagate more easily through spacetime.*

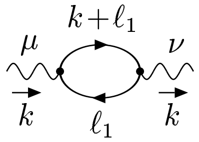
The physical picture behind this is that virtual fermion-antifermion pairs that are created from the vacuum partially screen the charges of the interacting particles (vacuum polarization), resulting in a lower effective charge. For larger $|k^2|$ more of the polarization cloud is penetrated and hence more of the actual charge can be felt.

All couplings in the Standard Model of electroweak interactions are in fact running couplings. As can be seen in the plot, the behaviour of the hypercharge coupling, indicated by U(1), resembles the one for QED. However, due to bosonic loop effects the couplings of the weak interactions, indicated by SU(2), and strong interactions, indicated by SU(3), actually become weaker for increasing invariant mass.



UV divergences: at 1-loop order the photon self-energy in QED is given by

$$i\Pi_2^{\mu\nu}(k) \;\equiv$$



$$= \;(-1)\,q^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \frac{\mathrm{Tr}(\gamma^\mu[\not{\ell}_1 + m]\gamma^\nu[\not{\ell}_1 + \not{k} + m])}{[(\ell_1+k)^2 - m^2 + i\epsilon][\ell_1^2 - m^2 + i\epsilon]}$$

$$= \;-4q^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \frac{\ell_1^\mu(\ell_1+k)^\nu + \ell_1^\nu(\ell_1+k)^\mu + g^{\mu\nu}[m^2 - \ell_1^2 - \ell_1\cdot k]}{[(\ell_1+k)^2 - m^2 + i\epsilon][\ell_1^2 - m^2 + i\epsilon]}$$

$$\overset{\text{p. 70}}{=\!=} \;-4q^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4} \int_0^1 \mathrm{d}\alpha_2\, \frac{2\,\ell_1^\mu\ell_1^\nu + \ell_1^\mu k^\nu + \ell_1^\nu k^\mu + g^{\mu\nu}[m^2 - \ell_1^2 - \ell_1\cdot k]}{(\ell_1^2 - m^2 + 2\alpha_2\ell_1\cdot k + \alpha_2 k^2 + i\epsilon)^2}$$

$$\overset{\ell = \ell_1 + \alpha_2 k}{=\!=} \;-4q^2 \int_0^1 \mathrm{d}\alpha_2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{2\,\ell^\mu\ell^\nu + g^{\mu\nu}(\Delta - \ell^2) + (k^2 g^{\mu\nu} - k^\mu k^\nu)2\alpha_2(1-\alpha_2)}{(\ell^2 - \Delta + i\epsilon)^2}$$

$$\overset{\text{p. 71}}{=\!=} \;-4q^2\, \frac{i}{16\pi^2} \int_0^1 \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\ell_E^2\, \ell_E^2\, \frac{g^{\mu\nu}(\Delta + \ell_E^2/2) + (k^2 g^{\mu\nu} - k^\mu k^\nu)2\alpha_2(1-\alpha_2)}{(\ell_E^2 + \Delta - i\epsilon)^2}\,,$$

where $\Delta = m^2 - \alpha_2(1-\alpha_2)k^2$. The resulting integral is clearly divergent.

Transversality lost: if we were to regularize (quantify) the UV divergence in the usual way by means of a cutoff Λ, then $\Pi_2^{\mu\nu}(k)$ would contain a leading singularity that is proportional to $g^{\mu\nu}\int_0^{\Lambda^2}\mathrm{d}\ell_E^2 = \Lambda^2 g^{\mu\nu}$. This has disastrous consequences, since it violates the transversality requirement and gives the photon an infinite mass. After all, a $\Lambda^2 g^{\mu\nu}$ term in $\Pi^{\mu\nu}(k)$ gives rise to a Λ^2/k^2 contribution to $\Pi(k^2)$ and therefore shifts the pole of $k^2\big[1 - \Pi(k^2)\big]$ away from $k^2 = 0$.

Question: what has happened here?

In fact the fermion-loop Ward – Takahashi identity on p. 131 has been invalidated, since we are actually not allowed to shift the integration variable without consequences when using the cutoff method.

14e *We need another regularization scheme that preserves the fundamental $U(1)$ symmetry, otherwise the results cannot be trusted. Dimensional regularization ('t Hooft – Veltman, 1972): compute Feynman diagrams as analytic functions of the dimensionality of spacetime. Use to this end an n-dimensional Minkowski space consisting of one time dimension and $n-1$ spatial dimensions.*

- *For sufficiently small n any loop integral will converge in the UV domain and the fermion-loop Ward – Takahashi identity is retained for all such n.*

- *The final expressions for observables are then obtained as* $n \to 4$ *limits.*

Examples of integrals calculated with dimensional regularization (DREG):

$$\int \frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^2} \xrightarrow{\text{DREG}} \int \frac{d^n\ell_E}{(2\pi)^n}\,\frac{1}{(\ell_E^2+\Delta)^2} \underline{\underline{\text{p. } 71, 72}} \; \frac{1}{(2\pi)^n}\,\frac{2\pi^{n/2}}{\Gamma(n/2)}\,\frac{1}{2}\int_0^\infty d\ell_E^2\,\frac{(\ell_E^2)^{n/2-1}}{(\ell_E^2+\Delta)^2}$$

$$\underline{\underline{z=\Delta/(\Delta+\ell_E^2)}} \; \frac{\Delta^{n/2-2}}{(4\pi)^{n/2}\,\Gamma(n/2)}\int_0^1 dz\, z^{1-n/2}(1-z)^{n/2-1} = \frac{\Delta^{n/2-2}}{(4\pi)^{n/2}}\,\Gamma(2-n/2)\ .$$

Here we have used the integral identity

$$\int_0^1 dz\, z^{b-1}(1-z)^{c-1} = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$$

in terms of the gamma function $\Gamma(z)$, which satisfies

$$\Gamma(1/2) = \sqrt{\pi} \quad , \quad \Gamma(1) = 1 \qquad \text{and} \qquad \Gamma(z+1) = z\Gamma(z)\ .$$

This time $\Gamma(2-n/2)$ represents the UV singularities, since the gamma function $\Gamma(z)$ has poles at $z = 0, -1, -2, \cdots$ and therefore $\Gamma(2-n/2)$ has poles at $n = \underline{4}, 6, 8, \cdots$:

$$\Gamma(2-n/2) \overset{n\approx 4}{\approx} -\frac{2}{n-4} - \gamma_E + \mathcal{O}(n-4) \quad \text{with} \quad \gamma_E = 0.5772 = \text{Euler's constant}\ .$$

In a similar way one finds

$$\int \frac{d^4\ell_E}{(2\pi)^4}\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} \xrightarrow{\text{DREG}} \frac{\Delta^{n/2-1}}{(4\pi)^{n/2}\,\Gamma(n/2)}\int_0^1 dz\, z^{-n/2}(1-z)^{n/2} = \frac{n\Delta}{2-n}\int \frac{d^n\ell_E}{(2\pi)^n}\,\frac{1}{(\ell_E^2+\Delta)^2}\ .$$

Transversality restored: returning to the integrand on page 134, we see that the non-transverse term indeed vanishes:

$$2\ell^\mu\ell^\nu + g^{\mu\nu}(\Delta-\ell^2) \xrightarrow{\text{DREG}} g^{\mu\nu}\big(\Delta - \ell^2[1-2/n]\big) \xrightarrow{\text{Wick}} g^{\mu\nu}\big(\Delta + \ell_E^2[1-2/n]\big) \xrightarrow{\text{integrals}} 0\ ,$$

as required by gauge symmetry. So, dimensional regularization is a viable way of dealing with UV divergences in the context of gauge symmetries. This regularization method was used successfully by 't Hooft and Veltman to prove the renormalizability of the Standard Model of electroweak interactions, for which they were awarded the Nobel Prize in 1999.