Exercise 20) Consider the electromagnetic Lagrangian density with 't Hooft-Feynman gauge-fixing term: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial.A)^2$, quantized by means of the canonical quantization condition $[\hat{A}^{\mu}(\vec{x},t), \hat{\pi}_{\nu}(\vec{y},t)] = i g^{\mu}{}_{\nu}\,\delta(\vec{x}-\vec{y})\,\hat{1}$.

(a) We want to implement the Lorenz condition without conflicting with the canonical quantization condition $[\hat{A}_0(\vec{x},t), \underbrace{\hat{\pi}_0(\vec{y},t)}_{-\partial.\hat{A}(\vec{y},t)}] = i\delta(\vec{x}-\vec{y})\,\hat{1}$

$\Rightarrow$ *) $\partial.\hat{A}(\vec{y},t) = 0$ would yield $[\hat{A}_0(\vec{x},t), \hat{\pi}_0(\vec{y},t)] = 0$ $\Rightarrow$ not okay!

*) $\partial.\hat{A}(\vec{y},t)|\psi\rangle = 0$ would imply $\langle\psi|[\hat{A}_0(\vec{x},t), \hat{\pi}_0(\vec{y},t)]|\psi\rangle = 0$ $\Rightarrow$ not okay!

(b) Try $\underline{\partial.\hat{A}^{(+)}|\psi\rangle = 0}$, with $\hat{A}^{(+)}_{\mu}(x) = \int\frac{d\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{r=0}^{3}\hat{a}^r_{\vec{p}}\,\epsilon^r_{\mu}(p)\,e^{-ip.x} = (\hat{A}^{(-)}_{\mu}(x))^{\dagger}$

$$\Rightarrow \langle\psi|\partial.\hat{A}|\psi\rangle = \langle\psi|\partial.\hat{A}^{(+)}|\psi\rangle + \langle\psi|\overbrace{\partial.\hat{A}^{(-)}}^{(\partial.\hat{A}^{(+)})^{\dagger}}|\psi\rangle = \langle\psi|\partial.\hat{A}^{(+)}|\psi\rangle + \langle\psi|\partial.\hat{A}^{(+)}|\psi\rangle^{*} = 0.$$

$\llcorner$ Lorenz condition implemented as an expectation-value condition

This is possible if $\partial.\hat{A}$ can be written as $\partial.\hat{A}^{(+)} + \partial.\hat{A}^{(-)}$, i.e. if $\partial.\hat{A}$ can be decomposed in plane-wave modes with accompanying creation and annihilation operators. This is okay if $\partial.\hat{A}$ satisfies the KG eqn. for massless particles (since $E_{\vec{p}} = |\vec{p}|$).

$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial.A)^2 \Rightarrow$ E.-L. eqn.: $-\Box A^{\nu} + (1-\lambda)\,\partial^{\nu}(\partial.A) = 0$

$\Rightarrow \partial_{\nu}(\text{E.-L. eqn.}) = -\lambda\,\Box(\partial.A) = 0$

So for a gauge-fixing term $-\frac{\lambda}{2}(\partial.A)^2$ ($\lambda \neq 0$, $\lambda \in \mathbb{R}$) it is indeed guaranteed that $\partial.A$ should satisfy the KG eqn.!

(c) $\partial.\hat{A}^{(+)}(x)|\psi\rangle = -i\int\frac{d\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{r=0}^{3} p.\epsilon^r(p)\,e^{-ip.x}|\psi\rangle = 0$, with

$\epsilon^{1,2}(p).p = 0$, $\epsilon^0(p).p = E_{\vec{p}}$, $\epsilon^3(p).p = -\vec{p}^{\,2}/E_{\vec{p}} = -E_{\vec{p}}$

$$\Rightarrow -i\int\frac{d\vec{p}}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\,e^{-ip.x}(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})|\psi\rangle = 0 \xrightarrow[\text{Fourier modes}]{e^{i\vec{p}.\vec{x}}\text{ bases of}} \boxed{(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})|\psi\rangle = 0}$$

for arbitrary photon momentum $\vec{p}$.

This means that physical states $|\psi\rangle$ in Fock space consist of the right admixture of scalar and longitudinal photons, whereas there is no restriction on the transverse-photon content!

Write $|\psi\rangle \equiv |\psi_T\rangle|\phi\rangle = |\psi_T\rangle\left[\,|\phi_0\rangle + c_1|\phi_1\rangle + c_2|\phi_2\rangle + \cdots\right]$ $\quad (c_1, c_2, \cdots \in \mathbb{C})$,

with $\hat a^r_{\vec p}|\psi_T\rangle = 0$ if $r = 0,3$ and $\langle\psi_T|\psi_T\rangle = 1$,

$\hat a^r_{\vec p}|\phi\rangle = 0$ if $r = 1,2$, $(\hat a^0_{\vec p} - \hat a^3_{\vec p})|\phi\rangle = 0$ and $\langle\phi_0|\phi_0\rangle = 1$.

(d) $\hat N_{SL} \equiv \int \frac{d\vec p}{(2\pi)^3}\left(\hat a^{3\dagger}_{\vec p}\hat a^3_{\vec p} - \hat a^{0\dagger}_{\vec p}\hat a^0_{\vec p}\right)$ counts scalar and longitudinal photons

↳ due to $[\hat a^0_{\vec p}, \hat a^{0\dagger}_{\vec p'}] = -(2\pi)^3\delta(\vec p - \vec p')\,\hat 1$

*) $$\hat N_{SL}|1_S\rangle = \int \frac{d\vec p\, d\vec p'}{(2\pi)^6}\frac{f(\vec p')}{\sqrt{2E_{\vec p'}}}\Big(\hat a^{3\dagger}_{\vec p}\underbrace{\cancel{\hat a^3_{\vec p}}}_{\to 0} - \hat a^{0\dagger}_{\vec p}\underbrace{\hat a^0_{\vec p}\,\hat a^{0\dagger}_{\vec p'}}_{\to\, -(2\pi)^3\delta(\vec p-\vec p')\hat 1}\Big)|0\rangle = |1_S\rangle$$

contains n scalar or long. photons

*) $\langle\phi_n|\hat N_{SL}|\phi_n\rangle = n\langle\phi_n|\phi_n\rangle$

$\Lsh$ $\int \frac{d\vec p}{(2\pi)^3}\langle\phi_n|(\hat a^{3\dagger}_{\vec p}\hat a^3_{\vec p} - \hat a^{0\dagger}_{\vec p}\hat a^0_{\vec p})|\phi_n\rangle = 0$, since $\hat a^3_{\vec p}|\phi_n\rangle = \hat a^0_{\vec p}|\phi_n\rangle$

$\Rightarrow\ n\langle\phi_n|\phi_n\rangle = 0 \ \Rightarrow\ \langle\phi_n|\phi_n\rangle = 0$ if $n \geq 1$

Consequence: $\langle\phi|\phi\rangle = \langle\phi_0|\phi_0\rangle + \sum_{n=1}^{\infty}|c_n|^2\langle\phi_n|\phi_n\rangle = 1$

(e) $$\langle\psi|N(\hat H)|\psi\rangle = \underbrace{\langle\phi|\phi\rangle}_{1}\langle\psi_T|\int\frac{d\vec p}{(2\pi)^3}E_{\vec p}\sum_{r=1}^{2}\hat a^{r\dagger}_{\vec p}\hat a^r_{\vec p}|\psi_T\rangle + \underbrace{\langle\psi_T|\psi_T\rangle}_{1}\langle\phi|\int\frac{d\vec p}{(2\pi)^3}E_{\vec p}\underbrace{\cancel{(\hat a^{3\dagger}_{\vec p}\hat a^3_{\vec p} - \hat a^{0\dagger}_{\vec p}\hat a^0_{\vec p})}}_{\to\,0\ \text{(as in part d)}}|\phi\rangle$$

<u>This means that the scalar and longitudinal photon contributions cancel each other in the (physical) energy expectation values !</u>

(f) $$\langle\psi|\hat A_\mu(x)|\psi\rangle = \underbrace{\langle\phi|\phi\rangle}_{1}\langle\psi_T|\hat A^{(tr)}_\mu(x)|\psi_T\rangle + \underbrace{\langle\psi_T|\psi_T\rangle}_{1}\langle\phi|\hat A_\mu(x)|\phi\rangle .$$

transverse modes only

In order to determine the last term we use the identity

$E_{\vec p}\,(\epsilon^0_\mu(p) + \epsilon^3_\mu(p)) = (E_{\vec p}, -\vec p) = p_\mu$, with $p^2 = 0$

$$\Rightarrow \langle\phi|\hat A_\mu(x)|\phi\rangle = \langle\phi|\left(\int\frac{d\vec p}{(2\pi)^3}\frac{e^{-ip.x}}{\sqrt{2E_{\vec p}}}\big[\epsilon^0_\mu(p)\,\hat a^0_{\vec p} + \epsilon^3_\mu(p)\,\hat a^3_{\vec p}\big] + \text{h.c.}\right)|\phi\rangle$$

↳ $\hat a^0_{\vec p}$ since $\hat a^3_{\vec p}|\phi\rangle = \hat a^0_{\vec p}|\phi\rangle$

$$= \langle\phi|\left(\int\frac{d\vec p}{(2\pi)^3}\,p_\mu\,\frac{e^{-ip.x}}{\sqrt{2E^3_{\vec p}}}\,\hat a^0_{\vec p} + \text{h.c.}\right)|\phi\rangle$$

$$= \partial_\mu\langle\phi|\left(\int\frac{d\vec p}{(2\pi)^3}\,\frac{i\,e^{-ip.x}}{\sqrt{2E^3_{\vec p}}}\,\hat a^0_{\vec p} + \text{h.c.}\right)|\phi\rangle \equiv \partial_\mu\chi(x)$$

$\Box\chi(x) = 0$, since $p^2 = 0$

cf. classical limit

(g) In terms of expectation values: $\langle\partial.\hat A\rangle = 0$ and $\langle\hat A_\mu\rangle = \langle\hat A^{(tr)}_\mu\rangle + \partial_\mu\chi$ $\quad(\Box\chi = 0)$.

The latter <u>reflects the remaining gauge arbitrariness</u> of $\langle\hat A_\mu\rangle$ <u>in the Lorenz gauge</u> $\Rightarrow$ $\chi(x)$ can be chosen to vanish, i.e. $|\phi\rangle = |\phi_0\rangle$.