Solution 14 (cont'd):

(c) Infinitesimal Lorentz transformation of a scalar field (see page 11 of the lecture notes):

$$\phi(x) \;\to\; \phi'(x) \;=\; \phi(\Lambda^{-1}x) \;\approx\; \big(1 - \frac{i}{2}\,\omega_{\mu\nu}J^{\mu\nu}\big)\phi(x)\,,$$

where the six generators of the Lorentz group are $J^{\mu\nu} = i\,(x^\mu\partial^\nu - x^\nu\partial^\mu)$.
They satisfy the fundamental commutation relations

$$[J^{\mu\nu},\, J^{\rho\sigma}] \;=\; -\,[x^\mu\partial^\nu - x^\nu\partial^\mu,\, x^\rho\partial^\sigma - x^\sigma\partial^\rho] \;=\; i\,(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})\,,$$

which can be verified by straightforward calculation.

(d) According to Dirac's trick, $S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu, \gamma^\nu]$ form a representation of $J^{\mu\nu}$:
by using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I_n$ one can rewrite $S^{\mu\nu} = \frac{i}{2}\,(\gamma^\mu\gamma^\nu - g^{\mu\nu}I_n) = -\,S^{\nu\mu}$ and obtain $[S^{\mu\nu},\, S^{\rho\sigma}] \;=\; -\,\frac{1}{4}\,[\,\gamma^\mu\gamma^\nu - g^{\mu\nu}I_n\,, \gamma^\rho\gamma^\sigma - g^{\rho\sigma}I_n\,] \;=\; -\,\frac{1}{4}\,[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma]$. With the help of

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \;&=\; -\,\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma + 2g^{\nu\rho}\gamma^\mu\gamma^\sigma \;=\; \ldots \\
&=\; +\,\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + 2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma\gamma^\nu - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu
\end{aligned}$$

one indeed arrives at the fundamental commutation relation

$$\begin{aligned}
[S^{\mu\nu},\, S^{\rho\sigma}] \;&=\; i\,(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\rho}S^{\sigma\nu} - g^{\nu\sigma}S^{\mu\rho} - g^{\mu\sigma}S^{\rho\nu}) \\
&=\; i\,(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})\,.
\end{aligned}$$

(e) The infinitesimal Lorentz transformations of four-vectors as derived in parts (a) and (b) can be written as

$$V^\alpha \;\to\; V'^{\,\alpha} \;=\; (g^\alpha_{\ \beta} + \omega^\alpha_{\ \beta})\,V^\beta \;=\; \big[\,g^\alpha_{\ \beta} \;-\; \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^\alpha_{\ \beta}\,\big]\,V^\beta\;,$$

with $(J^{\mu\nu})^\alpha_{\ \beta} \;=\; i\,(g^{\mu\alpha}g^\nu_\beta - g^\mu_\beta g^{\nu\alpha})$ the six generators of the Lorentz group for four-vectors. Indeed one finds:

$$\begin{aligned}
[J^{\mu\nu},\, J^{\rho\sigma}]^\alpha_{\ \beta} \;&=\; (J^{\mu\nu})^\alpha_{\ \gamma}\,(J^{\rho\sigma})^\gamma_{\ \beta} \;-\; (J^{\rho\sigma})^\alpha_{\ \gamma}\,(J^{\mu\nu})^\gamma_{\ \beta} \\
&=\; -\,(g^{\mu\alpha}g^\nu_\gamma - g^\mu_\gamma g^{\nu\alpha})\,(g^{\rho\gamma}g^\sigma_\beta - g^\rho_\beta g^{\sigma\gamma}) \;+\; (\mu\nu \leftrightarrow \rho\sigma) \\
&=\; i\,(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})^\alpha_{\ \beta}\,.
\end{aligned}$$

Solution 15:

The commutation relations for the generators of the Lorentz group are:

$$[J^{\mu\nu},\, J^{\rho\sigma}] \,=\, i\,(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})\,. \tag{1}$$

(a) Rotations: $L^j \,\equiv\, \frac{1}{2}\,\epsilon^{jkl}J^{kl}$ for $j=1,2,3$
with $J^{mn} = \frac{1}{2}\,(\delta^{mk}\delta^{nl} - \delta^{ml}\delta^{nk})J^{kl} = \frac{1}{2}\,\epsilon^{jmn}\epsilon^{jkl}J^{kl} = \epsilon^{mnj}L^j$.

Boosts: $K^j \,\equiv\, J^{0j}$ for $j=1,2,3$.

Infinitesimal Lorentz transformation for the field Φ: $\Phi \,\to\, \Phi' \,=\, (\,I - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K}\,)\Phi$.
We use eq. (1) to show that

★ $$[L^j, L^k] \,=\, \tfrac{1}{4}\,[\epsilon^{jmn}J^{mn},\, \epsilon^{krs}J^{rs}] \,=\, \tfrac{i}{4}\,\epsilon^{jmn}\epsilon^{krs}\,\big(-\delta^{nr}J^{ms} + \delta^{mr}J^{ns} \,-\, (r\leftrightarrow s)\big)$$
$$=\, -\tfrac{i}{2}\,\epsilon^{jmn}\,(\epsilon^{kns}J^{ms} - \epsilon^{kms}J^{ns}) \,=\, i\,\epsilon^{jmn}\epsilon^{kms}J^{ns} \,=\, i\,(\delta^{jk}\delta^{ns} - \delta^{js}\delta^{kn})J^{ns}$$
$$=\, -\,iJ^{kj} \,=\, iJ^{jk} \,=\, i\,\epsilon^{jkl}L^l.$$
$$[L^j, K^k] \,=\, \tfrac{1}{2}\,[\epsilon^{jmn}J^{mn},\, J^{0k}] \,=\, \tfrac{i}{2}\,\epsilon^{jmn}\,(\delta^{nk}J^{m0} - \delta^{mk}J^{n0}) \,=\, i\,\epsilon^{jmk}J^{m0} \,=\, i\,\epsilon^{jkl}K^l.$$
$$[K^j, K^k] \,=\, [J^{0j}, J^{0k}] \,=\, -\,iJ^{jk} \,=\, -\,i\,\epsilon^{jkl}L^l.$$

★ $$J^j_\pm \,\equiv\, \tfrac{1}{2}\,(L^j \pm iK^j)$$
$$\Rightarrow\;\; [J^j_\pm\,, J^k_\mp] \,=\, \tfrac{1}{4}\big(\,[L^j, L^k] \pm i\,[K^j, L^k] \mp i\,[L^j, K^k] + [K^j, K^k]\,\big) \,=\, 0 \;\text{ and}$$
$$[J^j_\pm\,, J^k_\pm] \,=\, \tfrac{1}{4}\big(\,[L^j, L^k] \pm i\,[K^j, L^k] \pm i\,[L^j, K^k] - [K^j, K^k]\,\big) \,=\, \tfrac{i}{2}\,\epsilon^{jkl}L^l \mp \tfrac{1}{2}\,\epsilon^{jkl}K^l$$
$$=\, i\,\epsilon^{jkl}J^l_\pm.$$

Hence, $\vec{J}_+$ commutes with $\vec{J}_-$ and they fulfill separately the angular momentum commutator algebra. The finite-dimensional irreducible representations of the Lorentz group are therefore labeled by a pair of integers or half integers (j_+, j_-), corresponding to pairs of representations of the rotation group with angular momentum quantum numbers $j_\pm$.

(b) $\Phi \,\to\, \Phi' \,=\, (\,I - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K}\,)\Phi \,=\, \big[\,I - (\vec{\beta} + i\vec{\theta}\,)\cdot\vec{J}_+ - (-\,\vec{\beta} + i\vec{\theta}\,)\cdot\vec{J}_-\,\big]\Phi\,.$

- $j_+ = \frac{1}{2}\,,\, j_- = 0$ representation: $\vec{J}_+ \,=\, \vec{\sigma}/2$ and $\vec{J}_- \,=\, \vec{0} \;\Rightarrow\; \Phi$ is a 2-component field with transformation property $\Phi \,\to\, \Phi' = (I_2 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2)\Phi$.

- $j_- = 0\,,\, j_+ = \frac{1}{2}$ representation: $\vec{J}_+ \,=\, \vec{0}$ and $\vec{J}_- \,=\, \vec{\sigma}/2 \;\Rightarrow\; \Phi$ is a 2-component field with transformation property $\Phi \,\to\, \Phi' = (I_2 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2)\Phi$.

Here we recognize the transformation properties of the two-dimensional left- and right-handed Weyl spinors, which represent the two two-dimensional irreducible representations of the Lorentz group. Both Weyl spinors can be combined to form the Dirac representation of the Lorentz group (which is not irreducible).

Solution 16:

Basic properties of the Dirac matrices γ^μ for $\mu = 0,1,2,3$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$:

1) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, especially $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = (\gamma^5)^2 = I_4$,

2) $\{\gamma^\mu, \gamma^5\} = 0$,

3) $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^5)^\dagger = \gamma^5$.

(a) Step 1: insert $I_4 = (\gamma^5)^2$; step 2: anticommute one of the (γ^5)'s to the other side of the trace, an odd number of anticommutations provides a '−' sign; step 3: use the cyclic property, i.e. $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$. All together

$$\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) \overset{\text{step 1}}{=\!=} \mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\gamma^5\gamma^5) \overset{\text{step 2}}{=\!=} -\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\gamma^5)$$

$$\overset{\text{step 3}}{=\!=} -\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\gamma^5\gamma^5) \overset{\text{eq.1)}}{=\!=} -\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}) = 0,$$

because the only complex number that is equal to its negative is 0.

Consequence: $\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\gamma^5) = 0$ since it also involves an odd number of γ-matrices.

(b) As in part (a), $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^0\gamma^0) = -\mathrm{Tr}(\gamma^0\gamma^5\gamma^0) = -\mathrm{Tr}(\gamma^5\gamma^0\gamma^0) = -\mathrm{Tr}(\gamma^5) = 0$ and $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = \pm\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5(\gamma^?)^2) = \mp\mathrm{Tr}(\gamma^?\gamma^\mu\gamma^\nu\gamma^5\gamma^?) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$, where by $\gamma^?$ we mean any of the γ-matrices that has an index that is different from μ and ν.

(c) Use here repeatedly $\gamma^{\mu_1}\gamma^{\mu_2} = \{\gamma^{\mu_1}, \gamma^{\mu_2}\} - \gamma^{\mu_2}\gamma^{\mu_1} = 2g^{\mu_1\mu_2} I_4 - \gamma^{\mu_2}\gamma^{\mu_1}$:

$$\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}) = 2g^{\mu_1\mu_2}\mathrm{Tr}(\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}) - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}) = \cdots =$$

$$2g^{\mu_1\mu_2}\mathrm{Tr}(\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}) - 2g^{\mu_1\mu_3}\mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_{2n}}) + \cdots - \mathrm{Tr}(\gamma^{\mu_2}\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}\gamma^{\mu_1})$$

$$\Rightarrow \quad \mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n}}) = \sum_{k=2}^{n}(-1)^k g^{\mu_1\mu_k}\mathrm{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}}\ldots\gamma^{\mu_{2n}})\,.$$

(d) Consequence: $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = g^{\mu\nu}\mathrm{Tr}(I_4) = 4g^{\mu\nu}$ as well as
$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = g^{\mu\nu}\mathrm{Tr}(\gamma^\rho\gamma^\sigma) - g^{\mu\rho}\mathrm{Tr}(\gamma^\nu\gamma^\sigma) + g^{\mu\sigma}\mathrm{Tr}(\gamma^\nu\gamma^\rho) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.

(e) Take $(\mu\nu\rho\sigma) \neq$ permutation of (0123), then two indices are equal and $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ can be written as a product of two Dirac matrices (e.g. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \pm\gamma^\rho\gamma^\sigma$ if $\mu = \nu$) . As $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$,

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 0 \quad \text{if} \quad (\mu\nu\rho\sigma) \neq \text{permutation of } (0123)\,.$$

If $(\mu\nu\rho\sigma) = (0123)$ then $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = \mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5) = -i\,\mathrm{Tr}(\gamma^5\gamma^5) = -4i$.
By permutation of these results one obtains

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma},$$

which can be obtained by interchanging the γ-matrices in order to bring them in the $\gamma^0\gamma^1\gamma^2\gamma^3$ order, producing a − sign at each interchange.