Solution 17:

Consider the free Dirac Lagrangian, which can be written in components as

$$\mathcal{L}(x) = \sum_{a,b} \psi_a^\star(x)\big[i(\gamma^0\gamma^\mu)_{ab}\partial_\mu - m(\gamma^0)_{ab}\big]\psi_b(x)\,.$$

(a) Euler-Lagrange equation for $\psi_a^\star(x)$:

$$\partial_\mu 0 - \sum_b \big[i(\gamma^0\gamma^\mu)_{ab}\partial_\mu - m(\gamma^0)_{ab}\big]\psi_b(x) = -\sum_c \gamma^0_{ac}\sum_b \big[i(\gamma^\mu)_{cb}\partial_\mu - m\delta_{cb}\big]\psi_b(x)$$

$$= -\sum_c \gamma^0_{ac}\big[(i\gamma^\mu\partial_\mu - m)\psi(x)\big]_c = 0 \quad\Rightarrow\quad \big[(i\gamma^\mu\partial_\mu - m)\psi(x)\big]_c = 0\,.$$

Euler-Lagrange equation for $\psi_b(x)$:

$$\partial_\mu \sum_a \psi_a^\star(x)\, i(\gamma^0\gamma^\mu)_{ab} + m\sum_a \psi_a^\star(x)(\gamma^0)_{ab} = \big[\bar\psi(x)(i\gamma^\mu \overleftarrow{\partial_\mu} + m)\big]_b = 0\,.$$

(b) As $\partial_\mu\alpha = 0$, the factors $e^{-i\alpha}$ transforming $\psi^\star$ and $e^{i\alpha}$ transforming ψ cancel each other in $\mathcal{L}$. So, the free Dirac Lagrangian has a symmetry under global phase transformations (= global $U(1)$ gauge transformations). This gives rise to the following conserved Noether current:

$$j^\mu(x) = \sum_b \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_b(x))}\Delta\psi_b(x) + 0 = \sum_{a,b}\psi_a^\star(x) i(\gamma^0\gamma^\mu)_{ab}\,[i\psi_b(x)] = -\,\bar\psi(x)\gamma^\mu\psi(x) = -\,j_V^\mu(x)\,,$$

which equals the Dirac vector current (up to a sign). Here we used that

$$\psi_b(x) \to e^{i\alpha}\psi_b(x) \approx \psi_b(x) + \alpha\,[i\psi_b(x)] \equiv \psi_b(x) + \alpha\Delta\psi_b(x)\,.$$

Since $\partial\mathcal{L}/\partial(\partial_\mu\psi_b(x))$ represents a row vector and $\Delta\psi_b(x)$ a column vector, we can derive the Noether current directly in spinor form:

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))}\Delta\psi(x) = i\bar\psi(x)\gamma^\mu[i\psi(x)] = -\,\bar\psi(x)\gamma^\mu\psi(x) = -\,j_V^\mu(x)\,.$$

(c) First: $\gamma^\mu e^{i\alpha\gamma^5} = \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\gamma^\mu(\gamma^5)^n \overset{\text{ex.16, eq.2}}{=\!=\!=} \sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}(-\gamma^5)^n\gamma^\mu = e^{-i\alpha\gamma^5}\gamma^\mu$ for $\mu = 0,1,2,3$. As $m = 0$ only the term with $\gamma^0\gamma^\mu$ in the Lagrangian remains. Commuting the γ_5-exponential with γ^0 gives one '-' sign in the exponent, with γ^μ again another one, and therefore the factors cancel as in case (b) above. Note that this is not the case for the mass term: it only involves γ^0 and hence flips the sign of the exponent. The additional Noether current for $m = 0$ reads

$$j^\mu(x) = \sum_b \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_b(x))}\Delta\psi_b(x) = \sum_{a,b}\psi_a^\star(x)\, i(\gamma^0\gamma^\mu)_{ab}\,[i\gamma^5\psi_b(x)] = -\,\bar\psi(x)\gamma^\mu\gamma^5\psi(x)\,,$$

which equals the Dirac axial vector current (up to a sign).

Polarisation sum $\sum_{s=1}^{2} u^s(p)\bar{u}^s(p)$: this 4x4 matrix

- depends on the 4-vector p^μ with $p^2=m^2$,
- has no dependence on a preferred spin vector (since we sum over the spin-½ spin basis),
- should have no open Minkowski indices,
- should yield 0 upon multiplication by $(\not{p}-m)$ from the left and right, since $(\not{p}-m)u^s(p) = \bar{u}^s(p)(\not{p}-m) = 0$.

Use the standard basis for 4x4 matrices without open $\mu\nu$ indices:

$$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = c_1 I_4 + c_{2\mu}\gamma^\mu + c_{3\mu\nu}\sigma^{\mu\nu} + c_{4\mu}\gamma^\mu\gamma^5 + c_5\gamma^5.$$

In the case of $c_{2\mu}$ and $c_{4\mu}$ the only vector at our disposal is p_μ and $c_{3\mu\nu}$ can only consist of the symmetric tensors $p_\mu p_\nu$ and $g_{\mu\nu}$, which both nullify the antisymmetric tensor $\sigma^{\mu\nu}$

$$\Rightarrow \sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = c_1 I_4 + c_2\not{p} + c_4\not{p}\gamma^5 + c_5\gamma^5.$$

Extra conditions:
$$0 = (\not{p}-m)\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = c_1(\not{p}-m) + c_2\overbrace{(\not{p}-m)\not{p}}^{m^2-m\not{p}} + c_4\underbrace{(\not{p}-m)\not{p}\gamma^5}_{m^2\gamma^5-m\not{p}\gamma^5} + c_5(\not{p}-m)\gamma^5$$
$$= (\not{p}-m)(c_1-mc_2) + (\not{p}-m)\gamma^5(c_5-mc_4) \quad \text{and}$$

$$0 = \sum_{s=1}^{2} u^s(p)\bar{u}^s(p)(\not{p}-m) = c_1(\not{p}-m) + c_2\overbrace{\not{p}(\not{p}-m)}^{m^2-m\not{p}} + c_4\underbrace{\not{p}\gamma^5(\not{p}-m)}_{-m^2\gamma^5-m\not{p}\gamma^5} + c_5\underbrace{\gamma^5(\not{p}-m)}_{-(\not{p}+m)\gamma^5}$$
$$= (\not{p}-m)(c_1-mc_2) - (\not{p}+m)\gamma^5(c_5+mc_4)$$

$\Rightarrow c_1 = mc_2$, $c_5 = mc_4 = -mc_4 \Rightarrow c_4 = c_5 = 0$, $c_1 = mc_2$,

hence $\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = c_2(\not{p}+m) \propto \not{p}+m$

depends on the normalization of the spinors, --- $c_2 = 1$ in the lecture notes

The discrete spacetime transformation of parity (spatial inversion):

$$x^\mu \xrightarrow{P} (\Lambda^P)^\mu{}_\nu x^\nu \quad \text{with} \quad (\Lambda^P)^\mu{}_\nu = \begin{pmatrix} 1 & & & \emptyset \\ & -1 & & \\ & & -1 & \\ \emptyset & & & -1 \end{pmatrix}.$$

Claim: in spinor space the parity transf. corresponds to $\Lambda_{1/2}^{(P)} = e^{i\varphi}\gamma^0 \quad (\varphi \in \mathbb{R})$.

(a) Let's verify that $\Lambda_{1/2}^{(P)-1}\gamma^\mu\Lambda_{1/2}^{(P)} = (\Lambda^P)^\mu{}_\nu\gamma^\nu$ in analogy with the continuous Lor. transf. case discussed in the lecture notes:

$\Lambda_{1/2}^{(P)-1} = e^{-i\varphi}\gamma^0$, since $e^{-i\varphi}\gamma^0 e^{+i\varphi}\gamma^0 = (\gamma^0)^2 = I_4$

$$\Rightarrow \Lambda_{1/2}^{(P)-1}\gamma^\mu\Lambda_{1/2}^{(P)} = \gamma^0\gamma^\mu\gamma^0 = \begin{cases} \gamma^0 & \text{if } \mu = 0 \\ -\gamma^j & \text{if } \mu = j = 1,2,3 \end{cases} = (\Lambda^P)^\mu{}_\nu\gamma^\nu. \quad \checkmark$$

(b) Under parity $\psi(x) \xrightarrow{P} \Lambda_{1/2}^{(P)}\psi(\tilde{x})$ with $\tilde{x}^\mu \equiv (x^0, -\vec{x})$

$$\Rightarrow \bar{\psi}(x) \xrightarrow{P} (\Lambda_{1/2}^{(P)}\psi(\tilde{x}))^\dagger\gamma^0 = \psi^\dagger(\tilde{x})\underbrace{\Lambda_{1/2}^{(P)\dagger}}_{= e^{-i\varphi}\gamma^0}\gamma^0 = \bar{\psi}(\tilde{x})e^{-i\varphi}\gamma^0 \overset{(a)}{=} \bar{\psi}(\tilde{x})\Lambda_{1/2}^{(P)-1}.$$

(c) Dirac currents: $\bar{\psi}(x)\Gamma\psi(x) \xrightarrow{P} \bar{\psi}(\tilde{x})\boxed{\gamma^0\Gamma\gamma^0}\psi(\tilde{x})$

$\to +I_4$ if $\Gamma = I_4$ (scalar current), $\quad -i\gamma^5$ if $\Gamma = i\gamma^5$ (pseudo scalar current) ①

$(\Lambda^P)^\mu{}_\nu\gamma^\nu$ if $\Gamma = \gamma^\mu$ (vector current), $\quad -(\Lambda^P)^\mu{}_\nu\gamma^\nu\gamma^5$ if $\Gamma = \gamma^\mu\gamma^5$ (axial vector current) ②

under a parity transf. the two currents in each of the sets ① and ② behave differently due to $\Lambda_{1/2}^{(P)-1}\gamma^5\Lambda_{1/2}^{(P)} = -\gamma^5 = \det(\Lambda^P)\gamma^5$.

Note: $\det(\Lambda) = -1$ for parity and $\det(\Lambda) = +1$ for cont. Lor. transf.

(d) $$\gamma^0 u^s(p) = \begin{pmatrix} \emptyset & I_2 \\ I_2 & \emptyset \end{pmatrix}\begin{pmatrix} \sqrt{p.\sigma}\,\xi^s \\ \sqrt{p.\bar{\sigma}}\,\xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{p.\bar{\sigma}}\,\xi^s \\ \sqrt{p.\sigma}\,\xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}.\sigma}\,\xi^s \\ \sqrt{\tilde{p}.\bar{\sigma}}\,\xi^s \end{pmatrix} = u^s(\tilde{p}),$$

with $\tilde{p}^\mu = (p^0, -\vec{p})$ and $p.\bar{\sigma} = p^0 I_2 + \vec{p}.\vec{\sigma} = \tilde{p}.\sigma$ etc.

$$\gamma^0 v^s(p) = \begin{pmatrix} \emptyset & I_2 \\ I_2 & \emptyset \end{pmatrix}\begin{pmatrix} \sqrt{p.\sigma}\,\eta^s \\ -\sqrt{p.\bar{\sigma}}\,\eta^s \end{pmatrix} = \begin{pmatrix} -\sqrt{p.\bar{\sigma}}\,\eta^s \\ \sqrt{p.\sigma}\,\eta^s \end{pmatrix} = -\begin{pmatrix} \sqrt{\tilde{p}.\sigma}\,\eta^s \\ -\sqrt{\tilde{p}.\bar{\sigma}}\,\eta^s \end{pmatrix} = -v^s(\tilde{p})$$

In the Dirac theory particles and antiparticles have opposite intrinsic parity!