Solution 20:

Notation: in $\hat{a}^s_{\vec{p}}$ the label s labels the spin quantum number and $\vec{p}$ the three-momentum. All fields used in this exercise are free fields, i.e. they would be interaction-picture fields in case of an interacting theory.

The positive frequency part $\hat{\psi}^+(x)$ contains $\hat{a}^s_{\vec{p}}$ and the negative frequency part $\hat{\psi}^-(x)$ contains $\hat{b}^{s\dagger}_{\vec{p}}$, whereas $\hat{\bar{\psi}}^+(x)$ contains $\hat{b}^s_{\vec{p}}$ and $\hat{\bar{\psi}}^-(x)$ contains $\hat{a}^{s\dagger}_{\vec{p}}$. Therefore, all anticommutators vanish except $\{\hat{\psi}^+_a(x), \hat{\bar{\psi}}^-_b(y)\}$ and $\{\hat{\bar{\psi}}^+_b(y), \hat{\psi}^-_a(x)\}$, where a and b are spinor indices.

$\boxed{x^0 > y^0}$:

$$
\begin{aligned}
T\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) &= \big(\hat{\psi}^+_a(x) + \hat{\psi}^-_a(x)\big)\big(\hat{\bar{\psi}}^+_b(y) + \hat{\bar{\psi}}^-_b(y)\big)\,, \\
N\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) &= \hat{\psi}^+_a(x)\hat{\bar{\psi}}^+_b(y) + \hat{\psi}^-_a(x)\big(\hat{\bar{\psi}}^+_b(y) + \hat{\bar{\psi}}^-_b(y)\big) - \hat{\bar{\psi}}^-_b(y)\hat{\psi}^+_a(x) \\
&= T\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) - \{\hat{\psi}^+_a(x), \hat{\bar{\psi}}^-_b(y)\}\,.
\end{aligned}
$$

$\boxed{x^0 < y^0}$:

$$
\begin{aligned}
T\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) &= -\big(\hat{\bar{\psi}}^+_b(y) + \hat{\bar{\psi}}^-_b(y)\big)\big(\hat{\psi}^+_a(x) + \hat{\psi}^-_a(x)\big)\,, \\
N\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) &= -N\big(\hat{\bar{\psi}}_b(y)\hat{\psi}_a(x)\big) \;=\; T\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) + \{\hat{\bar{\psi}}^+_b(y), \hat{\psi}^-_a(x)\}\,.
\end{aligned}
$$

Hence, $T\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) = N\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) + \overbracket{\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)} = N\big(\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)\big) + [S_F(x-y)]_{ab}\,\hat{1}\,.$

Since $\{\hat{\psi}^+_a(x), \hat{\psi}^-_b(y)\} = \{\hat{\bar{\psi}}^+_a(x), \hat{\bar{\psi}}^-_b(y)\} = 0$, one has $\overbracket{\hat{\psi}_a(x)\hat{\psi}_b(y)} = \overbracket{\hat{\bar{\psi}}_a(x)\hat{\bar{\psi}}_b(y)} = 0\,.$

The proof of Wick's theorem follows the steps outlined on p. 38 and 39 of the lecture notes with $\hat{\phi}_j$ representing a fermionic field at the spacetime point x_j, i.e. either $\hat{\phi}_j = \hat{\psi}_{a_j}(x_j)$ or $\hat{\phi}_j = \hat{\bar{\psi}}_{a_j}(x_j)$. Since interchanging two fields now generates a minus sign, the differences to the scalar case are

$$
\begin{aligned}
\hat{\phi}^+_1 N(\hat{\phi}_2\cdots\hat{\phi}_m) &= N\big(\{\hat{\phi}^+_1, \hat{\phi}^-_2\}\hat{\phi}_3\cdots\hat{\phi}_m - \hat{\phi}_2\{\hat{\phi}^+_1, \hat{\phi}^-_3\}\hat{\phi}_4\cdots\hat{\phi}_m + \cdots \\
&\qquad + (-1)^{m-2}\,\hat{\phi}_2\cdots\hat{\phi}_{m-1}\{\hat{\phi}^+_1, \hat{\phi}^-_m\}\big) + (-1)^{m-1}N(\hat{\phi}_2\cdots\hat{\phi}_m)\,\hat{\phi}^+_1 \\
&= N(\hat{\phi}^+_1\hat{\phi}_2\cdots\hat{\phi}_m + \overbracket{\hat{\phi}_1\hat{\phi}_2}\hat{\phi}_3\cdots\hat{\phi}_m + \overbracket{\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3}\hat{\phi}_4\cdots\hat{\phi}_m + \cdots)\,,
\end{aligned}
$$

where we have used that $N(\hat{\phi}_2\cdots\hat{\phi}_m)\,\hat{\phi}^+_1 = (-1)^{m-1}N(\hat{\phi}^+_1\hat{\phi}_2\cdots\hat{\phi}_m)\,.$

For the four-point Green's function this implies:

$$
\begin{aligned}
&\langle 0|T\big(\hat{\psi}_{a_1}(x_1)\hat{\psi}_{a_2}(x_2)\hat{\bar{\psi}}_{a_3}(x_3)\hat{\bar{\psi}}_{a_4}(x_4)\big)|0\rangle \\
&= \langle 0|N\big(\hat{\psi}_{a_1}(x_1)\hat{\psi}_{a_2}(x_2)\hat{\bar{\psi}}_{a_3}(x_3)\hat{\bar{\psi}}_{a_4}(x_4) + \text{all possible contractions}\big)|0\rangle \\
&= [S_F(x_1-x_4)]_{a_1a_4}\,[S_F(x_2-x_3)]_{a_2a_3} - [S_F(x_1-x_3)]_{a_1a_3}\,[S_F(x_2-x_4)]_{a_2a_4}\,,
\end{aligned}
$$

where only fully contracted terms contribute and the minus sign originates from Fermi statistics.

Solution 21:

Let's consider a fermionic theory with Lagrangian

$$\mathcal{L}(x) \;=\; \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m_\psi)\psi(x) \,+\, \frac{1}{2}\left[\partial_\mu\phi(x)\right]\left[\partial^\mu\phi(x)\right] \,-\, \frac{1}{2}m_\phi^2\phi^2(x) \,-\, g\,\bar{\psi}(x)\Gamma\psi(x)\phi(x)\;,$$

with Γ a specific 4×4 matrix, $\phi(x)$ a real scalar field and $\psi(x)$ a Dirac field. This gives rise to the following interaction term in the Hamilton operator: $\hat{H}_{\text{int}} = g\int \mathrm{d}\vec{x}\;\hat{\bar{\psi}}(x)\,\Gamma\,\hat{\psi}(x)\hat{\phi}(x)$.

(a) We first derive the Euler-Lagrange equations for all three fields:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} \;=\; \partial_\mu(i\bar{\psi}\gamma^\mu) + m_\psi\bar{\psi} + g\,\bar{\psi}\,\Gamma\phi \;=\; \bar{\psi}(i\overleftarrow{\not{\partial}} + m_\psi + g\,\phi\Gamma) \;=\; 0\;,$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \;=\; \partial_\mu(0) - (i\not{\partial} - m_\psi)\psi + g\,\Gamma\psi\phi \;=\; -\,(i\not{\partial} - m_\psi - g\,\phi\Gamma)\psi \;=\; 0\;,$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} \;=\; \partial_\mu(\partial^\mu\phi) + m_\phi^2\phi + g\,\bar{\psi}\,\Gamma\psi \;=\; (\Box + m_\phi^2)\,\phi + g\,\bar{\psi}\,\Gamma\psi \;=\; 0\;.$$

(b) We know that the action has mass dimension 0, i.e. $[S] = 0$. Therefore we have

$$\left[\int \mathrm{d}^4x\,\mathcal{L}\right] = 0 \quad\Rightarrow\quad [\mathcal{L}] \;=\; 4\;.$$

Furthermore, we know that

$$[m_\psi] \;=\; [m_\phi] \;=\; [\partial_\mu] \;=\; 1\;.$$

This leads to the following mass dimensions for the remaining objects:

$$[\phi] \;=\; \frac{4-2}{2} \;=\; 1\;\;,\;\; [\psi] \;=\; [\bar{\psi}] \;=\; \frac{4-1}{2} \;=\; \frac{3}{2} \qquad\text{and}\qquad [g] \;=\; 4 - 2\cdot\frac{3}{2} - 1 \;=\; 0\;.$$

Hence, g is a dimensionless coupling constant in four spacetime dimensions! So, we anticipate to be dealing with a renormalizable theory.

(c) The non-interacting (interaction picture) field $\hat{\psi}_I(x)$ contains the operators $\hat{a}_{\vec{p}}, \hat{b}^\dagger_{\vec{p}}$, while $\hat{\bar{\psi}}_I(x)$ contains the operators $\hat{a}^\dagger_{\vec{p}}, \hat{b}_{\vec{p}}$. This implies that

$$\langle 0|\,T\big(\hat{\psi}_{a_I}(x)\,\hat{\psi}_{b_I}(y)\big)\,|0\rangle \;=\; \langle 0|\,T\big(\hat{\bar{\psi}}_{a_I}(x)\,\hat{\bar{\psi}}_{b_I}(y)\big)\,|0\rangle \;=\; 0\;,$$

since in both cases the two sets of operators contained in the two fields anticommute and therefore annihilate the vacuum $|0\rangle$ or $\langle 0|$.

(d) Next we want to determine

$$\langle 0|\,T\Big(\hat{\psi}_{a_I}(x_1)\,\hat{\bar{\psi}}_{b_I}(x_2)\,\hat{\phi}_I(x_3)\,\mathrm{e}^{-i\int \mathrm{d}^4z\;\hat{\mathcal{H}}_{\text{int}_I}(z)}\Big)\,|0\rangle$$

to first order in the coupling constant g. Expanding the time-ordered product up to first order in the interaction and applying Wick's theorem, we find

$$
\begin{aligned}
&\langle 0|\, T\Big(\hat\psi_{a_I}(x_1)\,\hat{\bar\psi}_{b_I}(x_2)\,\hat\phi_I(x_3)\,\mathrm{e}^{-ig\int \mathrm{d}^4z\,\sum_{c,d}\hat{\bar\psi}_{c_I}(z)\,\Gamma_{cd}\,\hat\psi_{d_I}(z)\hat\phi_I(z)}\Big)|0\rangle \\
\stackrel{\mathcal{O}(g)}{=\!=}\;& \langle 0|\, T\left(\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\Big[\hat 1 - ig\sum_{c,d}\int \mathrm{d}^4z\,\hat{\bar\psi}_{c_I}(z)\,\Gamma_{cd}\,\hat\psi_{d_I}(z)\hat\phi_I(z)\Big]\right)|0\rangle \\
\stackrel{\text{Wick}}{=\!=}\;& -ig\sum_{c,d}\int \mathrm{d}^4z\,\langle 0|\,\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\hat{\bar\psi}_{c_I}(z)\,\Gamma_{cd}\,\hat\psi_{d_I}(z)\hat\phi_I(z)\,|0\rangle \qquad 1:- \quad 2:+ \\
& -ig\sum_{c,d}\int \mathrm{d}^4z\,\langle 0|\,\overbrace{\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)}^{1:+}\underbrace{\hat\phi_I(x_3)\overbrace{\hat{\bar\psi}_{c_I}(z)\,\Gamma_{cd}\,\hat\psi_{d_I}(z)}^{2:+}\hat\phi_I(z)}\,|0\rangle\ .
\end{aligned}
$$

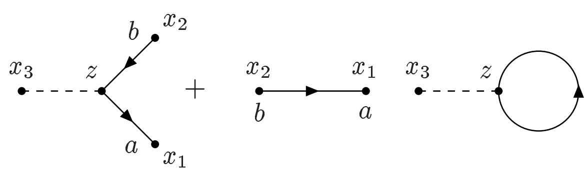
Note that the lowest-order term in the expansion vanishes, because it is a product of three fields and therefore cannot be fully contracted. Furthermore, we used that contractions of $\hat\psi$ with $\hat\psi$ vanish, just like contractions of $\hat{\bar\psi}$ with $\hat{\bar\psi}$ (in accordance with part d).

Next, we should replace contractions by propagators. The fermionic propagator is defined as the following contraction:

$$
[S_F(x-y)]_{ab} \;=\; \hat\psi_{a_I}(x)\hat{\bar\psi}_{b_I}(y) \;=\; -\,\hat{\bar\psi}_{b_I}(y)\hat\psi_{a_I}(x)\ .
$$

Therefore, the propagator picks up an additional minus sign if the order of the two fields is interchanged. Altogether, we find for the $\mathcal{O}(g)$ terms:

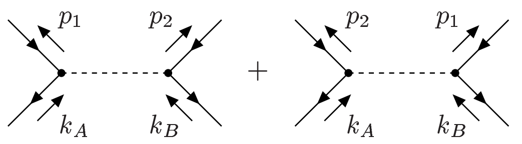
$$
\begin{aligned}
&+ig\int \mathrm{d}^4z\, D_F(x_3-z)\sum_{c,d}\,[S_F(x_1-z)]_{ac}\,\Gamma_{cd}\,[-S_F(z-x_2)]_{db} \\
&-ig\,[S_F(x_1-x_2)]_{ab}\int \mathrm{d}^4z\, D_F(x_3-z)\sum_{c,d}\Gamma_{cd}\,[-S_F(z-z)]_{dc} \\
=\;& -ig\int \mathrm{d}^4z\, D_F(x_3-z)\,[S_F(x_1-z)\,\Gamma\, S_F(z-x_2)]_{ab} \\
&+ig\,[S_F(x_1-x_2)]_{ab}\int \mathrm{d}^4z\, D_F(x_3-z)\,\mathrm{Tr}\big(\Gamma S_F(z-z)\big)\ .
\end{aligned}
$$

Diagrammatically this can be represented by 

Note that the analytic expressions nicely confirm the Feynman rules for fermion loops, for the vertex in the Yukawa theory and for the various aspects of the arrow convention!

(e+f) The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\bar\psi(k_A,s_A)\bar\psi(k_B,s_B)\;\to\;\bar\psi(p_1,r_1)\bar\psi(p_2,r_2)\big)$ is given by

$$i\mathcal{M} \;=$$



$$\equiv \; i\mathcal{M}_1 + i\mathcal{M}_2$$

$$= \; -(-ig)^2\, i \left(\frac{[\bar{v}^{s_A}(k_A)\,\Gamma\, v^{r_1}(p_1)]\,[\bar{v}^{s_B}(k_B)\,\Gamma\, v^{r_2}(p_2)]}{(k_A-p_1)^2 - m_\phi^2 + i\epsilon} \;-\; (p_1,r_1) \leftrightarrow (p_2,r_2) \right) \;.$$

The first of these contributions corresponds to the contractions

$$1:+ \qquad 4:+$$

$${}_0\langle \vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\, \hat{\bar{\psi}}_I(x)\,\Gamma\,\hat{\psi}_I(x)\hat{\phi}_I(x)\int \mathrm{d}^4y\, \hat{\bar{\psi}}_I(y)\,\Gamma\,\hat{\psi}_I(y)\hat{\phi}_I(y)\,|\vec{k}_A\vec{k}_B\rangle_0$$

$$2:- \qquad 3:+$$

and the second one to the contractions

$$2:+ \qquad 4:+$$

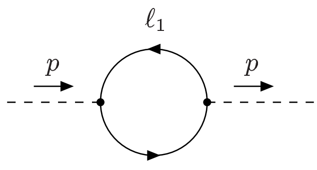
$${}_0\langle \vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\, \hat{\bar{\psi}}_I(x)\,\Gamma\,\hat{\psi}_I(x)\hat{\phi}_I(x)\int \mathrm{d}^4y\, \hat{\bar{\psi}}_I(y)\,\Gamma\,\hat{\psi}_I(y)\hat{\phi}_I(y)\,|\vec{k}_A\vec{k}_B\rangle_0 \;.$$

$$1:+ \qquad 3:+$$

As expected from Fermi statistics, the two diagrams have a relative minus sign as a result of the interchange of the two identical final-state antifermions.

(g) For $\Gamma = I_4$ the one-loop contribution to the self-energy of a scalar boson with arbitrary momentum p is given as follows:

$$-\,i\,\Sigma_\phi(p^2) \;\overset{\text{one-loop}}{=\!=}$$



$$= \; -(-ig)^2\, i^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \frac{\mathrm{Tr}\big(\,[\not{\ell}_1 + m_\psi]\,[\not{\ell}_1 + \not{p} + m_\psi]\,\big)}{[\ell_1^2 - m_\psi^2 + i\epsilon]\,[(\ell_1+p)^2 - m_\psi^2 + i\epsilon]} \;.$$

Note that the first minus sign and the trace appear because we are dealing with a fermion loop. We can now evaluate the trace with the identities that we found in exercise 16. Recall that the trace over an odd number of gamma matrices vanishes, and that

$$\mathrm{Tr}(I_4) \;=\; 4 \quad \text{and} \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) \;=\; 4g^{\mu\nu} \;.$$

We then find for the trace

$$\mathrm{Tr}\Big(\not{\ell}_1(\not{\ell}_1 + \not{p}) + m_\psi(2\not{\ell}_1 + \not{p}) + m_\psi^2\Big) \;=\; 4\big[\,\ell_1\cdot(\ell_1+p) + m_\psi^2\,\big] \;,$$

giving us the desired result.

(h) Consider an arbitrary 1-particle irreducible amputated loop diagram in the Yukawa theory with N_F external fermions, N_B external bosons, P_F fermion propagators, P_B boson propagators, V vertices, and L loop momenta:

- two fermions and one boson meet in each vertex, each external line is connected to one vertex and each propagator is connected to two vertices or to the same vertex twice:

$$2V = N_F + 2P_F \quad , \quad V = N_B + 2P_B \ ,$$

which implies that N_F is always even and

$$P_F = V - \frac{1}{2}N_F \quad , \quad P_B = \frac{1}{2}(V - N_B) \ ;$$

- as usual $L = P - V + 1$, with P the total number of propagators, i.e.

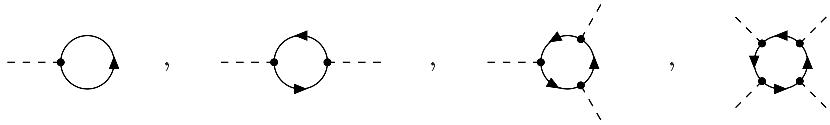
$$L = (P_F + P_B) - V + 1 = \frac{1}{2}(V - N_F - N_B) + 1 \quad \Rightarrow \quad L \geq 1 \quad \text{if} \quad V \geq N_B + N_F \ .$$

Naive power counting tells us that each loop momentum yields Λ^4, each boson propagator yields Λ^{-2}, and each fermion propagator Λ^{-1}, as long as we are working in four spacetime dimensions. Therefore the superficial degree of divergence of the considered loop diagram equals

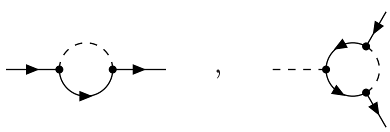
$$D = 4L - 2P_B - P_F = 4 - \frac{3}{2}N_F - N_B \ .$$

Since D is independent of V, divergences (i.e. $D \geq 0$) can occur at all loop orders, but there is only a finite number of divergent amplitudes (i.e. amplitudes with $N_F = 0\,, N_B \leq 4$ or $N_F = 2\,, N_B \leq 1$). As anticipated, the Yukawa theory is indeed renormalizable in four spacetime dimensions!

(i) The superficially divergent 1-particle irreducible one-loop diagrams can be divided into two categories. The one involving a fermion loop and one, two, three or four external boson lines:



and the one with two external fermion lines and either zero or one external boson line:



The first divergence can be absorbed by shifting the scalar field. The self-energy diagrams in both sets can be renormalized by applying mass and wave-function renormalization to the free scalar/Dirac parts of the Langrangian. The divergence in the last diagram can be absorbed into the coupling constant g. The odd ones out are the last two diagrams in the first set. In order to renormalize the associated divergences explicit triple and quartic scalar interactions have to be added as counterterms to the Lagrangian, which suffices to absorb all divergences!