Exercise 22) Use the trace technology of exercise 16.

(a) $(\bar v(k')\gamma^\nu u(k))^* = (v^\dagger(k')\gamma^0\gamma^\nu u(k))^* = u^\dagger(k)\gamma^{\nu\dagger}\gamma^{0\dagger}v(k') \overset{\text{Ex.16}}{=} u^\dagger(k)\gamma^0\gamma^\nu\gamma^0\gamma^0 v(k')$

$= \bar u(k)\gamma^\nu v(k')\,,$

$(\bar v(k')\gamma^\nu\gamma^5 u(k))^* = (v^\dagger(k')\gamma^0\gamma^\nu\gamma^5 u(k))^* = u^\dagger(k)\gamma^{5\dagger}\gamma^{\nu\dagger}\gamma^{0\dagger}v(k') \overset{\text{Ex.16}}{=} u^\dagger(k)\gamma^5\gamma^0\gamma^\nu\gamma^0\gamma^0 v(k')$

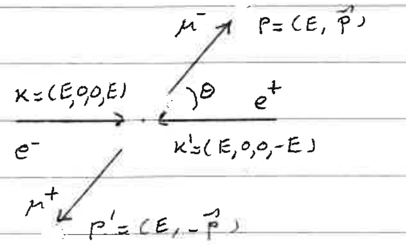
$= -u^\dagger(k)\gamma^0\gamma^5\gamma^\nu v(k') = -\bar u(k)\gamma^5\gamma^\nu v(k')\,.$

(b) Use that traces with an odd number of γ-matrices vanish:

$$\frac14\sum_{pol.}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\underbrace{Tr([\not k'-m_e]\gamma^\rho[\not k+m_e]\gamma^\nu)}_{4(k'^\rho k^\nu+k^\rho k'^\nu-g^{\rho\nu}[k.k'+m_e^2])}\ \underbrace{Tr([\not p+m_\mu]\gamma_\rho[\not p'-m_\mu]\gamma_\nu)}_{4(p_\rho p'_\nu+p'_\rho p_\nu-g_{\rho\nu}[p.p'+m_\mu^2])}$$

$$= \frac{8e^4}{q^4}\left[(p'.k)(p.k)+(p'.k')(p.k)+m_e^2\,p.p'+m_\mu^2\,k.k'+2m_e^2m_\mu^2\right].$$

(c) Neglecting me the following holds in the CM frame:



$E = \frac12 E_{CM}$

$p \equiv |\vec p| = \sqrt{E^2-m_\mu^2} \geq 0 \Rightarrow \boxed{E\geq m_\mu}$!

$\vec p\cdot\vec e_z = p\cos\theta$

$q^2 = (p+p')^2 = (k+k')^2 = 4E^2$

$k.k' = 2E^2$

$p.k = p'.k' = E^2 - Ep\cos\theta = E(E-p\cos\theta) > 0$

$p.k' = p'.k = E^2 + Ep\cos\theta = E(E+p\cos\theta) > 0$

Hence, $\frac14\sum_{pol.}|\mathcal{M}|^2 \overset{(b)}{=} \frac{e^4}{2E^4}\left[E^2(E+p\cos\theta)^2+E^2(E-p\cos\theta)^2+2m_\mu^2E^2\right]$

$\overset{p=\sqrt{E^2-m_\mu^2}}{=} e^4\left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$

(d) $\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{\Theta(E-m_\mu)}{64\pi^2E_{CM}^2}\frac{|\vec p|}{|\vec k|}\frac14\sum_{pol}|\mathcal{M}|^2 \overset{e^2=4\pi\alpha}{=} \Theta(E-m_\mu)\frac{\alpha^2}{16E^2}\sqrt{1-m_\mu^2/E^2}\left[1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$

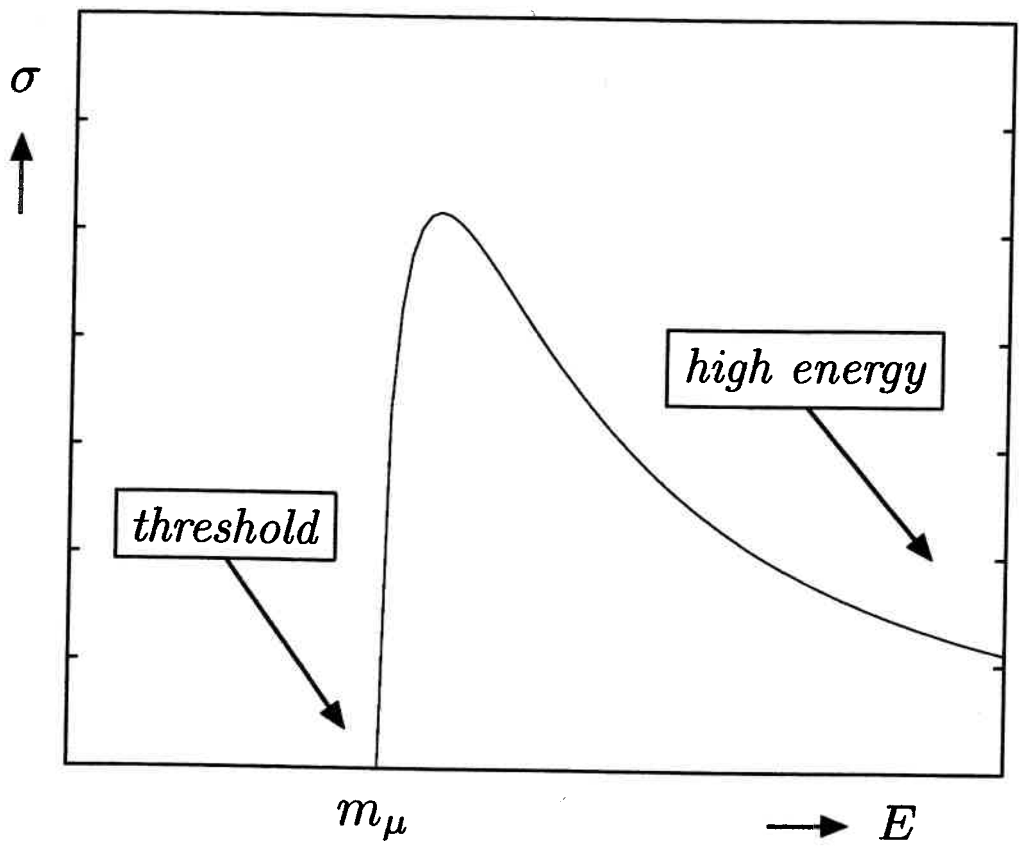
$\Rightarrow \sigma_{tot}^{unpol} = \Theta(E-m_\mu)\frac{\alpha^2}{16E^2}\sqrt{1-m_\mu^2/E^2}\left[2\pi * \frac83 + \frac{m_\mu^2}{E^2}2\pi * \frac43\right]$

$= \boxed{\Theta(E-m_\mu)\frac{\pi\alpha^2}{3E^2}\sqrt{1-m_\mu^2/E^2}\left[1+\frac{m_\mu^2}{2E^2}\right] = \sigma_{tot}^{unpol}}\,.$

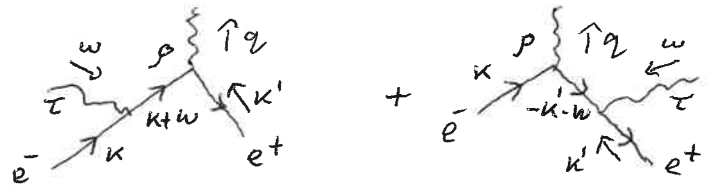
At high energies $(E \gg m_\mu)$: $\sigma_{tot}^{unpol} \approx \frac{\pi\alpha^2}{3E^2}$.

Near threshold $(E \approx m_\mu)$: $\sigma_{tot}^{unpol} \approx \frac{\pi\alpha^2}{2m_\mu^2}\Theta(E-m_\mu)\sqrt{1-m_\mu^2/E^2}$.

Sketch of the energy dependence of the cross section $\sigma_{tot}^{unpol.}$



(e) There are two places where we can attach the photon:



$$\left(\ \cdots \ + \ \cdots \ \right) * \omega_\tau$$

$(-k\!\!\!/' - m_e) - (-k\!\!\!/' - \omega\!\!\!/ - m_e)$

$$= i|e|\,\bar v(k')\left[\gamma^\rho \frac{i(k\!\!\!/+\omega\!\!\!/+m_e)}{(k+\omega)^2 - m_e^2}\, i|e|\,\omega\!\!\!/ + i|e|\,\omega\!\!\!/\, \frac{i(-k\!\!\!/'-\omega\!\!\!/+m_e)}{(k'+\omega)^2 - m_e^2}\,\gamma^\rho\right] u(k)$$

Dirac eqns.

$(k\!\!\!/+\omega\!\!\!/-m_e) - (k\!\!\!/-m_e)$

$$\underline{\underline{(k\!\!\!/-m_e)u(k)=0}}$$

$$\bar v(k')(k\!\!\!/'+m_e)=0$$

$$i|e|\,\bar v(k')\left[\gamma^\rho i \cdot i|e| - i \cdot i|e|\,\gamma^\rho\right] u(k) = 0 .$$

as expected from the WI for on-shell amplitudes