Ex. 29) Quantized electromagnetic field inside a finite cubic enclosure with periodic boundary conditions:

$$\hat{\vec{A}}(\vec r,t) = \frac{1}{\sqrt{\epsilon_0}}\sum_{\vec k}\sum_{\lambda=1}^{2} \underbrace{\vec u_{\vec k,\lambda}(\vec r)}_{\propto\, e^{i\vec k\cdot\vec r}\,\vec\varepsilon_\lambda(\vec e_k)} \sqrt{\frac{\hbar}{2\omega_k}}\left[\hat a_{\vec k,\lambda}(t) + \eta_\lambda \hat a^\dagger_{-\vec k,\lambda}(t)\right], \quad \dot{\hat a}_{\vec k,\lambda}(t) = -i\omega_k \hat a_{\vec k,\lambda}(t) \quad (1).$$

Contribution to the Hamilton operator originating from the quantized $\hat{\vec{\mathcal{E}}}$-field:

$$\frac{\epsilon_0}{2}\int_V d\vec r\; \hat{\vec{\mathcal{E}}}(\vec r,t)\cdot\hat{\vec{\mathcal{E}}}^\dagger(\vec r,t) = \frac{\epsilon_0}{2}\int_V d\vec r\; \frac{\partial\hat{\vec A}}{\partial t}(\vec r,t)\cdot\frac{\partial\hat{\vec A}^\dagger}{\partial t}(\vec r,t)$$

$$\underset{(1)}{=} \frac{1}{2}\sum_{\vec k,\vec k'}\sum_{\lambda,\lambda'=1}^{2}\int_V d\vec r\; \vec u_{\vec k,\lambda}(\vec r)\cdot\vec u^*_{\vec k',\lambda'}(\vec r)\,\frac{\hbar}{2}\sqrt{\omega_k\omega_{k'}}\left[\hat a_{\vec k,\lambda}(t)-\eta_\lambda\hat a^\dagger_{-\vec k,\lambda}(t)\right]\left[\hat a^\dagger_{\vec k',\lambda'}(t)-\eta_{\lambda'}\hat a_{-\vec k',\lambda'}(t)\right]$$

$$\underset{(E.5)}{=} \frac{1}{4}\sum_{\vec k}\sum_{\lambda=1}^{2}\hbar\omega_k\left[\hat a_{\vec k,\lambda}(t)-\eta_\lambda\hat a^\dagger_{-\vec k,\lambda}(t)\right]\left[\hat a^\dagger_{\vec k,\lambda}(t)-\eta_\lambda\hat a_{-\vec k,\lambda}(t)\right] \quad (2)$$

Contribution to the Hamilton operator originating from the quantized $\hat{\vec{B}}$-field:

$$\frac{\epsilon_0 c^2}{2}\int_V d\vec r\; \hat{\vec B}(\vec r,t)\cdot\hat{\vec B}^\dagger(\vec r,t) = \frac{\epsilon_0 c^2}{2}\int_V d\vec r\; (\vec\nabla\times\hat{\vec A}(\vec r,t))\cdot(\vec\nabla\times\hat{\vec A}^\dagger(\vec r,t))$$

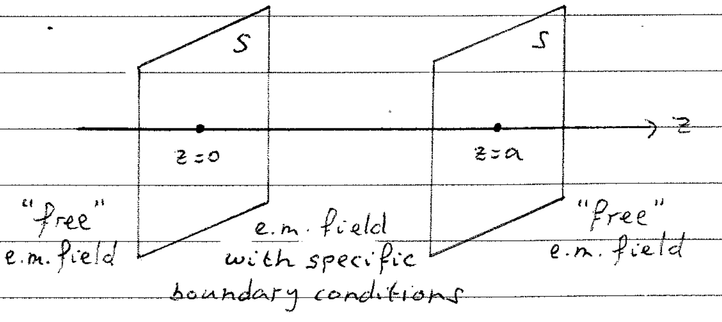
$$\underset{ck=\omega_k}{\overset{(1)}{=}} \frac{1}{2}\sum_{\vec k,\vec k'}\sum_{\lambda,\lambda'=1}^{2}\int_V d\vec r\; (\vec e_k\times\vec u_{\vec k,\lambda}(\vec r))\cdot(\vec e_{k'}\times\vec u^*_{\vec k',\lambda'}(\vec r))\,\frac{\hbar}{2}\sqrt{\omega_k\omega_{k'}}\left[\hat a_{\vec k,\lambda}(t)+\eta_\lambda\hat a^\dagger_{-\vec k,\lambda}(t)\right]\left[\hat a^\dagger_{\vec k',\lambda'}(t)+\eta_{\lambda'}\hat a_{-\vec k',\lambda'}(t)\right]$$

$$\underset{(E.6)}{=} \frac{1}{4}\sum_{\vec k}\sum_{\lambda=1}^{2}\hbar\omega_k\left[\hat a_{\vec k,\lambda}(t)+\eta_\lambda\hat a^\dagger_{-\vec k,\lambda}(t)\right]\left[\hat a^\dagger_{\vec k,\lambda}(t)+\eta_\lambda\hat a_{-\vec k,\lambda}(t)\right] \quad (3)$$

$$\Rightarrow \hat H = (2)+(3) = \underbrace{\sum_{\vec k}\sum_{\lambda=1}^{2}}_{\text{symmetric under } \vec k\to-\vec k}\frac{1}{2}\hbar\omega_k\Big(\underbrace{\hat a_{\vec k,\lambda}(t)\hat a^\dagger_{\vec k,\lambda}(t)}_{\text{comm. relations}:\ \hat n_{\vec k,\lambda}+\hat 1} + \underbrace{\hat a^\dagger_{-\vec k,\lambda}(t)\hat a_{-\vec k,\lambda}(t)}_{\hat n_{-\vec k,\lambda}}\Big) \underset{\vec k\to-\vec k}{=} \boxed{\sum_{\vec k}\sum_{\lambda=1}^{2}\hbar\omega_k\left(\hat n_{\vec k,\lambda}+\tfrac{1}{2}\hat 1\right) = \hat H}$$

spectrum $\left\{\sum_{\vec k}\sum_{\lambda}\hbar\omega_k(n_{\vec k,\lambda}+\frac{1}{2}) : n_{\vec k,\lambda}\in\mathbb{N}\right\}$ bounded from below by $\sum_{\vec k}\sum_{\lambda}\frac{1}{2}\hbar\omega_k$

Ex. 30)



S large, perfectly conducting plates: ($\sqrt{S}\gg a$)

$$\vec{\mathcal{E}}_{\parallel}(\vec r_{\parallel}, z=0, t) = \vec{\mathcal{E}}_{\parallel}(\vec r_{\parallel}, z=a, t) = \vec 0$$

$$B_z(\vec r_{\parallel}, z=0, t) = B_z(\vec r_{\parallel}, z=a, t) = 0$$

$\vec r_{\parallel}$: coordinates $\parallel$ plates, i.e. in the xy-plane

(i) $\vec{A}_{\parallel}(\vec{r}_{\parallel}, z=0, t) = \vec{A}_{\parallel}(\vec{r}_{\parallel}, z=a, t) = \vec{0}$, with $\vec{E} = -\partial\vec{A}/\partial t$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ in the Coulomb gauge $\Rightarrow$ *) $\vec{E}_{\parallel} = \vec{0}$ on the plates, since $\vec{A}_{\parallel} = \vec{0}$ on the plates for all times t,

*) $B_z = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = 0$ on the plates, since $\vec{A}_{\parallel} = \vec{0}$ and thus $A_x = A_y = 0$ for all coordinates on the plates.

(ii) $\vec{A}_{\parallel} = \vec{0}$ on the plates $\Rightarrow$ $\vec{A}_{\parallel}$ can be decomposed in terms of Fourier modes that have a half integer or integer number of oscillations between $z=0$ and $z=a$. Between the plates $\vec{A}_{\parallel}$ coincides with the vector potential corresponding to a periodic e.m. field with period $2a$ in the z direction. Since $\partial A_z/\partial z \overset{\text{Coulomb cond.}}{=} -\vec{\nabla}_{\parallel}\cdot\vec{A}_{\parallel}$, the same will hold for the z-component A_z.

(iii) Quantized e.m. field with periodic boundary condition (period $2a$) in the z direction and a continuum limit in the $\vec{r}_{\parallel}$ directions; k_z is quantized according to $k_z = \frac{2\pi}{L}\nu \overset{L=2a}{=} \nu\pi/a$ $(\nu \in \mathbb{Z})$, $\vec{k}_{\parallel}$ takes on continuous values with $\int d\vec{k}_{\parallel}/(4\pi^2)$ states on the interval $[\vec{k}_{\parallel}, \vec{k}_{\parallel} + d\vec{k}_{\parallel}]$ (see eq. (355) in the lecture notes)

$\Rightarrow$ $\omega_{\kappa} = ck = c\sqrt{\vec{k}_{\parallel}^2 + k_z^2} = c\sqrt{\vec{k}_{\parallel}^2 + (\nu\pi/a)^2}$.

Consequence: zero-point energy of the periodic e.m. field inside the volume $2Sa$

$$\int d\vec{k}_{\parallel} = \int_0^{2\pi} d\varphi \int_0^{\infty} d|\vec{k}_{\parallel}|\,|\vec{k}_{\parallel}|$$

$$E_0(2Sa) = \frac{S}{4\pi^2}\int d\vec{k}_{\parallel} \sum_{\nu}\sum_{\lambda=1}^{2} \frac{1}{2}\hbar\omega_{\kappa} = \frac{\hbar c S}{4\pi}\int_0^{\infty} d\vec{k}_{\parallel}^2 \sum_{\nu=-\infty}^{\infty} \sqrt{\vec{k}_{\parallel}^2 + (\nu\pi/a)^2}.$$

(iv) Free e.m. field: $\sum_{\nu} \rightarrow \frac{\overset{L}{(2a)}}{2\pi}\int_{-\infty}^{\infty} dk_z \equiv \int_{-\infty}^{\infty} d\nu \Rightarrow E_0^{free}(2Sa) = \frac{\hbar c S}{4\pi}\int_0^{\infty} d\vec{k}_{\parallel}^2 \int_{-\infty}^{\infty} d\nu \sqrt{\vec{k}_{\parallel}^2 + (\nu\pi/a)^2}$.

int. instead of sum

(v) Renormalized zero-point energy for the e.m. field between the plates:

$E_0^{ren}(a) = \underbrace{\tfrac{1}{2}E_0(2Sa)}_{\infty} - \underbrace{\tfrac{1}{2}E_0^{free}(2Sa)}_{\infty} = -\frac{\pi^2}{720}\frac{\hbar c S}{a^3}$ after a tedious calculation,

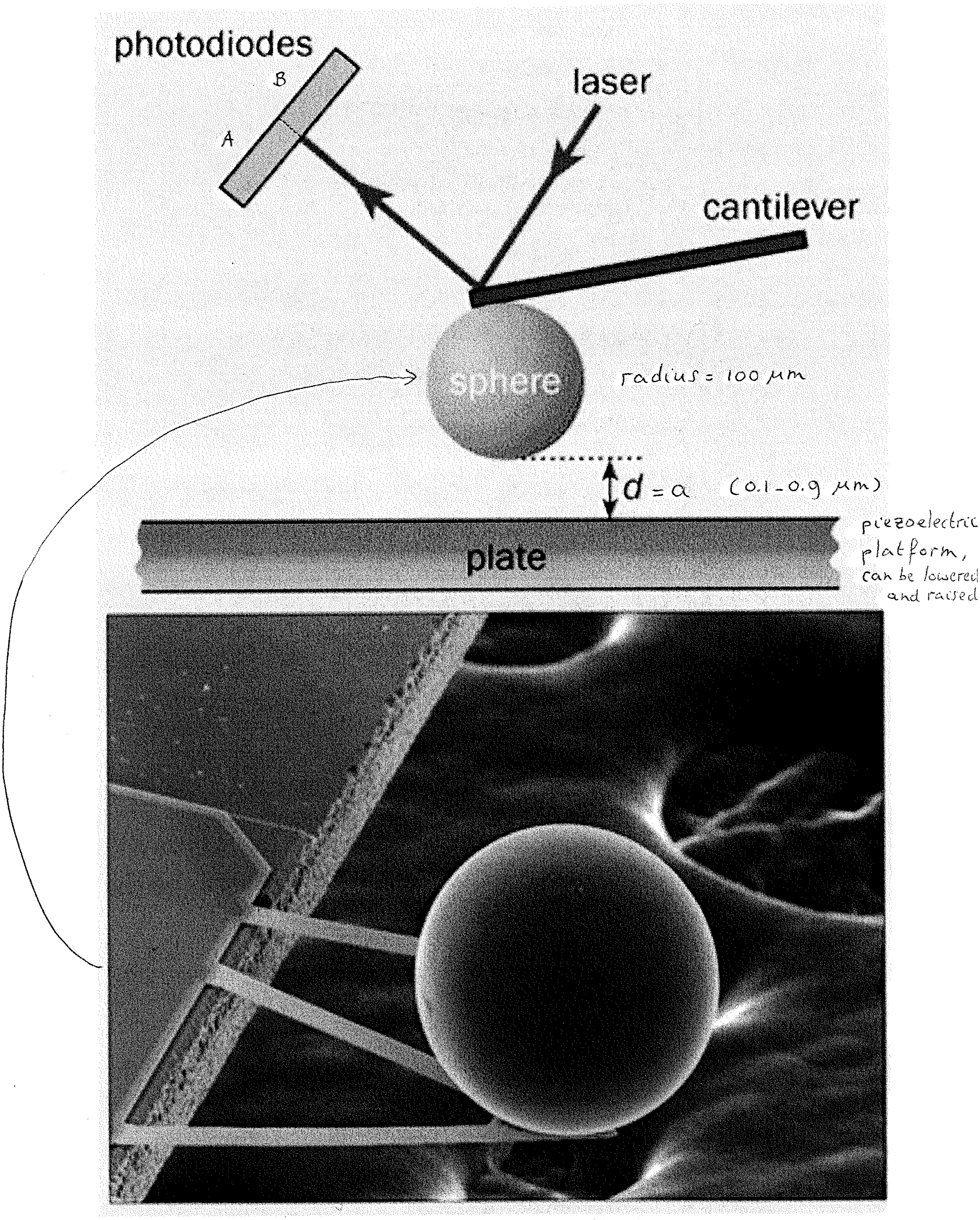
$F_0^{ren}(a) = -\frac{\partial}{\partial a}E_0^{ren}(a) = -\frac{\pi^2}{240}\frac{\hbar c S}{a^4}$ ← attractive force between the plates (Casimir effect)

(vi) Power counting: $\overset{\text{"dimension" of pressure } P}{[P]} = [F/S] = kg\, m^{-1} s^{-2}$, $\underbrace{[\hbar] = kg\, m^2 s^{-1}, [c] = m\, s^{-1}, [a] = m}_{\text{finite system parameters}}$

$P \equiv const.\, \hbar^{\gamma} c^{\beta} a^{\sigma} \Rightarrow \gamma = 1, \beta = 1, \sigma = -4$.

On dimensional grounds we thus expect that $P = const.\, \hbar c/a^4$!

Mohideen & Roy (1998) : Casimir effect



Experimental confirmation of the Casimir effect