Solution 1:

(a) $L = T - V$ with (i) kinetic term $T = \sum_{n=1}^{N} \frac{1}{2} m\dot{\phi}_n^2(t)$, as the kinetic energy of all point particles simply adds up, and (ii) elastic term $V = \sum_{n=1}^{N} \frac{1}{2} k_s(\phi_{n+1} - \phi_n)^2$, as the potential energy of all springs add up and $|\phi_{n+1} - \phi_n|$ is identical to the elongation/compression of the spring, i.e., the deviation from the equilibrium length.

(b)

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_n} - \frac{\partial L}{\partial \phi_n} = m\ddot{\phi}_n + k_s(2\phi_n - \phi_{n-1} - \phi_{n+1})\,,$$

because

$$\begin{aligned}\frac{\partial}{\partial \phi_n}\sum_{j=1}^{N}\frac{1}{2}k_s(\phi_{j+1} - \phi_j)^2 &= \sum_{j=1}^{N}\frac{1}{2}k_s(2\phi_{j+1}\delta_{n,j+1} - 2\phi_j\delta_{n,j+1} - 2\phi_{j+1}\delta_{n,j} + 2\phi_j\delta_{n,j}) \\ &= \frac{1}{2}k_s(2\phi_n - 2\phi_{n-1} - 2\phi_{n+1} + 2\phi_n)\,.\end{aligned}$$

(c) a becomes so small that $\sum_{n=1}^{N} \to \frac{1}{a}\int_0^{L=Na} dx$. Thus

$$L = \frac{1}{a}\int_0^L dx\,\left(\frac{1}{2}m(\sqrt{a}\dot{\phi}(x,t))^2 - \frac{1}{2}k_s(\sqrt{a}\phi(x+a,t) - \sqrt{a}\phi(x,t))^2\right)\,.$$

Employing the Taylor expansion $\phi(x \pm a, t) = \phi(x,t) \pm a\phi'(x,t) + \frac{1}{2}a^2\phi''(x,t) + \ldots$ one gets

$$L = \int_0^L dx\mathcal{L}, \quad \mathcal{L} = \frac{1}{2}m\dot{\phi}^2(x,t) - \frac{1}{2}k_s a^2(\phi'(x,t))^2\,.$$

The Taylor expansion is also used for the equation of motion (e.o.m.):

$$\begin{aligned}0 &= m\sqrt{a}\ddot{\phi}(x,t) + k_s\sqrt{a}(2\phi(x,t) - \phi(x-a,t) - \phi(x+a,t)) \\ &= \sqrt{a}\left(m\ddot{\phi}(x,t) + k_s(2\phi(x,t) - \phi(x,t) + a\phi'(x,t) - \frac{1}{2}a^2\phi''(x,t) - \phi(x,t) - a\phi'(x,t) - \frac{1}{2}a^2\phi''(x,t))\right) \\ &= \sqrt{a}\left(m\ddot{\phi}(x,t) - k_s a^2\phi''(x,t)\right)\,.\end{aligned}$$

In terms of $\partial_t \equiv \partial/\partial t$ and $\partial_x \equiv \partial/\partial x$ this becomes:

$$\mathcal{L}(\phi, \partial_t\phi, \partial_x\phi) = \frac{1}{2}m\left(\partial_t\phi\right)^2 - \frac{1}{2}k_s a^2\left(\partial_x\phi\right)^2 \qquad \text{and} \qquad m\left(\partial_t^2 - \frac{k_s a^2}{m}\partial_x^2\right)\phi(x,t) = 0\,.$$

(d) In the Euler-Lagrange equation $\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} + \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$ the last term vanishes, the first two terms give directly

$$m\partial_t^2\phi(x,t) - k_s a^2\partial_x^2\phi(x,t) = 0\,.$$

(e) The solutions are of the form $\phi_+(x + v_p t) + \phi_-(x - v_p t)$, i.e. phonons (sound waves) moving to the left or right with speed $v_p = a\sqrt{k_s/m}$.

(f) In mechanics one has the Hamiltonian $H(p,q) = p\dot{q} - L$ with the momentum $p = \frac{\partial L}{\partial \dot{q}}$, in field theory the Hamiltonian density $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ with the momentum field $\pi = \partial\mathcal{L}/\partial\dot{\phi}$.
Here one has $\pi(x,t) = m\dot{\phi}(x,t)$ and therefore

$$\mathcal{H}(\phi, \pi, \partial_x\phi) = m\dot{\phi}^2 - \frac{1}{2}m\dot{\phi}^2 + \frac{1}{2}k_s a^2(\partial_x\phi)^2 = \frac{\pi^2}{2m} + \frac{1}{2}mv_p^2\left(\partial_x\phi\right)^2\,.$$

Solution 2:

The action of electrodynamics is given by $S = \int d^4x\, \mathcal{L} = \int d^4x\, \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)$ with the anti-symmetric rank-2 tensor (field strength tensor) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that this tensor (and thus the action) depends only on the derivative of the gauge field and not on the gauge field directly, i.e., $\partial\mathcal{L}/\partial A_\mu = 0$. Furthermore, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is the derivative w.r.t. the contravariant coordinate vector x^μ and is (in flat space-time) a covariant vector. It has the properties $\partial_0 = \partial/\partial t$ and $\partial_i = \partial/\partial x^i = \nabla^i$ for $i = 1, 2, 3$.

The Euler-Lagrange equation reads therefore

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} - \frac{\partial \mathcal{L}}{\partial A_\beta} = \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = 0\,.$$

First we write $F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ and note that

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} &= -\frac{1}{4}\left(\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right) F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - \frac{1}{4} F_{\mu\nu}\left(\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)\right) g^{\mu\rho} g^{\nu\sigma} \\ = -\frac{1}{4}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right) F^{\mu\nu} - \frac{1}{4} F^{\rho\sigma}\left(\delta^\alpha_\rho \delta^\beta_\sigma - \delta^\alpha_\sigma \delta^\beta_\rho\right) &= -\frac{1}{4}\left(F^{\alpha\beta} - F^{\beta\alpha}\right) - \frac{1}{4}\left(F^{\alpha\beta} - F^{\beta\alpha}\right) = -F^{\alpha\beta}\,, \end{aligned} \tag{1}$$

where $F^{\alpha\beta} = -F^{\beta\alpha}$ has been used. This then leads to the compact Euler-Lagrange equation

$$\partial_\alpha F^{\alpha\beta} = 0\,. \tag{2}$$

This equation is a four-vector equation. To decompose it into temporal and spatial components we note first that

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$

i.e., we have $F^{00} = 0$, $F^{0i} = -E^i$, $F^{ij} = -\epsilon^{ijk}B^k$. Now we set $\beta = 0$ in eq. (2) to obtain Gauss' law:

$$0 = \partial_\alpha F^{\alpha 0} = \partial_i F^{i0} = \partial_i E^i = \vec{\nabla}\cdot\vec{E}\,,$$

where $i = 1, 2, 3$ has been summed over. Setting now $\beta = j = 1, 2, 3$ in eq. (2) gives

$$0 = -\partial_\alpha F^{\alpha j} = -\partial_0 F^{0j} - \partial_i F^{ij} = \partial_0 E^j + \partial_i \epsilon^{ijk} B^k = \frac{\partial E^j}{\partial t} - (\vec{\nabla}\times\vec{B})^j\,.$$

The Maxwell equations $\vec{\nabla}\cdot\vec{B} = 0$ and $\partial\vec{B}/\partial t + \vec{\nabla}\times\vec{E} = \vec{0}$ can be derived directly from the definitions and the antisymmetry of ϵ^{ijk}:

$$\vec{\nabla}\cdot\vec{B} = \partial_k B^k = -\frac{1}{2}\partial_k \epsilon^{ijk} F^{ij} = \frac{1}{2}\epsilon^{ijk}(\partial^k\partial^i A^j - \partial^k\partial^j A^i) = 0\,,$$

$$\frac{\partial B^k}{\partial t} + (\vec{\nabla}\times\vec{E})^k = -\frac{1}{2}\epsilon^{ijk}\partial^0 F^{ij} - \epsilon^{ijk}\partial_i F^{0j} = \epsilon^{ijk}\left(\frac{1}{2}\partial^0\partial^j A^i + \frac{1}{2}\partial^0\partial^i A^j - \partial^i\partial^j A^0\right) = 0\,.$$

Alternatively this can be written as

$$\partial_\alpha \tilde{F}^{\alpha\beta} \equiv \partial_\alpha \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}(\partial_\alpha\partial_\mu A_\nu - \partial_\alpha\partial_\nu A_\mu) = 0\,,$$

in terms of the totally antisymmetric tensor $\epsilon^{\alpha\beta\mu\nu}$, whose non-zero components are given by $+1/-1$ for $(\alpha\beta\mu\nu)$ being an even/odd permutation of (0123). The tensor $\tilde{F}^{\alpha\beta}$ is called the dual field strength tensor.