Solution 2 (cont'd):

The improved electromagnetic energy-momentum tensor reads: $\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu)$.
First we note that the expression for the energy-momentum tensor for a scalar field,

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \mathcal{L}\,g^{\mu\nu}\,,$$

is generalized to

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\sigma)}\partial^\nu A_\sigma - \mathcal{L}\,g^{\mu\nu}$$

for a vector field. From last week's part of this exercise we know that $\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\sigma)} = -F^{\mu\sigma}$, resulting in

$$T^{\mu\nu} = -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}\,.$$

With the help of

$$\partial_\lambda(F^{\mu\lambda}A^\nu) = (\partial_\lambda F^{\mu\lambda})A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \overset{\text{week 1}}{=\!=\!=} 0 + F^{\mu\lambda}(\partial_\lambda A^\nu)$$

and

$$-F^{\mu\sigma}\partial^\nu A_\sigma + F^{\mu\lambda}(\partial_\lambda A^\nu) = F^{\mu\lambda}(\partial_\lambda A^\nu - \partial^\nu A_\lambda) = F^{\mu\lambda}F_\lambda{}^\nu$$

one eventually obtains

$$\hat{T}^{\mu\nu} = F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}\,,$$

which is indeed symmetric under $\mu \leftrightarrow \nu$. Therefore, summing over repeated indices we get

$$\begin{aligned}\mathcal{E} &= \hat{T}^{00} = F^{0\lambda}F_\lambda{}^0 + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{00} = F^{0i}F^{0i} - \frac{1}{2}F^{0i}F^{0i}g^{00} + \frac{1}{4}F^{ij}F^{ij}g^{00} \\ &= E^iE^i - \frac{1}{2}E^iE^i + \frac{1}{4}\epsilon^{ijk}B^k\epsilon^{ijl}B^l \overset{\text{hint week 1}}{=\!=\!=\!=} \vec{E}^2 - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\end{aligned}$$

for the energy density carried by the electromagnetic field, and

$$S^i = \hat{T}^{0i} = \hat{T}^{i0} = F^{0\lambda}F_\lambda{}^i + 0 = F^{0j}F_j{}^i = -F^{0j}F^{ji} = E^j(-\epsilon^{jik}B^k) = \epsilon^{ijk}E^jB^k = (\vec{E}\times\vec{B})^i$$

for the momentum density (which is also known as the Poynting vector).

Solution 3:

$$\mathcal{L} = (\partial_\mu\phi_1^*)(\partial^\mu\phi_1) + (\partial_\mu\phi_2^*)(\partial^\mu\phi_2) - m^2(\phi_1^*\phi_1 + \phi_2^*\phi_2)$$

(a) The equations of motion for $\phi_{1,2}(x)$ and $\phi_{1,2}^*(x)$ are of the Klein-Gordon type:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_{1,2})} - \frac{\partial\mathcal{L}}{\partial\phi_{1,2}} = 0 \quad\Longrightarrow\quad \partial_\mu\partial^\mu\phi_{1,2}^* + m^2\phi_{1,2}^* = 0\,,$$

and

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_{1,2}^*)} - \frac{\partial\mathcal{L}}{\partial\phi_{1,2}^*} = 0 \quad\Longrightarrow\quad \partial_\mu\partial^\mu\phi_{1,2} + m^2\phi_{1,2} = 0\,.$$

(b) One obtains now four conjugate momentum fields $\pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$:

$$\pi_{\phi_{1,2}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_{1,2})} = \partial^0\phi_{1,2}^* \equiv \pi_{1,2} \quad\text{and}\quad \pi_{\phi_{1,2}^*} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_{1,2}^*)} = \partial^0\phi_{1,2} \equiv \pi_{1,2}^*\,.$$

(c) The Hamiltonian is the integral over the Hamiltonian density, i.e. $H = \int d^3x\, \mathcal{H}$, where

$$\mathcal{H} = \pi_1 \partial_0 \phi_1 + \pi_2 \partial_0 \phi_2 + \pi_1^* \partial_0 \phi_1^* + \pi_2^* \partial_0 \phi_2^* - \mathcal{L} = 2\pi_1^* \pi_1 + 2\pi_2^* \pi_2 - \mathcal{L}\,.$$

Since the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \pi_1^* \pi_1 + \pi_2^* \pi_2 - (\vec{\nabla}\phi_1) \cdot (\vec{\nabla}\phi_1^*) - (\vec{\nabla}\phi_2) \cdot (\vec{\nabla}\phi_2^*) - m^2 (\phi_1^* \phi_1 + \phi_2^* \phi_2)$$

one has

$$\mathcal{H} = \pi_1^* \pi_1 + \pi_2^* \pi_2 + (\vec{\nabla}\phi_1) \cdot (\vec{\nabla}\phi_1^*) + (\vec{\nabla}\phi_2) \cdot (\vec{\nabla}\phi_2^*) + m^2 (\phi_1^* \phi_1 + \phi_2^* \phi_2)\,,$$

consisting of kinetic terms, elastic terms and rest mass terms.

(d) Now we introduce the vector (doublet) notation $\vec{\Phi} \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ for the two complex scalar fields and write compactly $\mathcal{L} = (\partial_\mu \vec{\Phi}^\dagger)(\partial^\mu \vec{\Phi}) - m^2 \vec{\Phi}^\dagger \vec{\Phi}$. Under the continuous global $U(1)$ transformation

$$\vec{\Phi}(x) \to e^{i\alpha} \vec{\Phi}(x) \qquad \text{and} \qquad \vec{\Phi}^\dagger(x) \to e^{-i\alpha} \vec{\Phi}^\dagger(x) \qquad (\alpha \in \mathbb{R})$$

the Lagrangian is invariant:

$$\mathcal{L} \to (\partial_\mu \vec{\Phi}^\dagger) e^{-i\alpha} e^{i\alpha} (\partial^\mu \vec{\Phi}) - m^2 \vec{\Phi}^\dagger e^{-i\alpha} e^{i\alpha} \vec{\Phi} = \mathcal{L}\,.$$

Keeping a close eye on the order of the vectors, the corresponding Noether current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\Phi})} \Delta\vec{\Phi} + \Delta\vec{\Phi}^\dagger \frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\Phi}^\dagger)} = (\partial^\mu \vec{\Phi}^\dagger) i \vec{\Phi} + (-i\vec{\Phi}^\dagger)(\partial^\mu \vec{\Phi}) = i \left[(\partial^\mu \vec{\Phi}^\dagger)\vec{\Phi} - \vec{\Phi}^\dagger (\partial^\mu \vec{\Phi}) \right]$$

using the first-order variations[1] $\Delta\vec{\Phi} = i\vec{\Phi}$ and $\Delta\vec{\Phi}^\dagger = -i\vec{\Phi}^\dagger$. The corresponding Noether charge is

$$\int d^3x\, j^0 = i \int d^3x \left[(\partial^0 \vec{\Phi}^\dagger)\vec{\Phi} - \vec{\Phi}^\dagger (\partial^0 \vec{\Phi}) \right] \overset{(b)}{=\!=} i \sum_{a=1}^{2} \int d^3x \left[\pi_a \phi_a - \phi_a^* \pi_a^* \right].$$

(e) Under the continuous global $SU(2)$ transformation

$$\vec{\Phi}(x) \to e^{i\alpha^k \sigma^k} \vec{\Phi}(x) \qquad \text{and} \qquad \vec{\Phi}^\dagger(x) \to \vec{\Phi}^\dagger(x) e^{-i\alpha^k \sigma^k} \qquad (\alpha^k \in \mathbb{R} \ \text{ for } \ k = 1,2,3)$$

the Lagrangian is also invariant, which can be shown by an analogous calculation as above. Here the hermitian Pauli matrices $\sigma^{1,2,3}$ are the generators of $SU(2)$. With

$$(\Delta\vec{\Phi})^k = i\sigma^k \vec{\Phi} \qquad \text{and} \qquad (\Delta\vec{\Phi}^\dagger)^k = -i\vec{\Phi}^\dagger \sigma^k\,,$$

the corresponding three conserved Noether currents and charges read

$$(j^\mu)^k = (\partial^\mu \vec{\Phi}^\dagger) i\sigma^k \vec{\Phi} - i\vec{\Phi}^\dagger \sigma^k (\partial^\mu \vec{\Phi})\,,$$

$$Q^k \equiv \int d^3x\, (j^0)^k = i \int d^3x \left[(\partial^0 \vec{\Phi}^\dagger)\sigma^k \vec{\Phi} - \vec{\Phi}^\dagger \sigma^k (\partial^0 \vec{\Phi}) \right] = i \sum_{a,b=1}^{2} \int d^3x \left[\pi_a (\sigma^k)_{ab}\, \phi_b - \phi_a^* (\sigma^k)_{ab}\, \pi_b^* \right].$$

[1]By definition one has $\vec{\Phi}' = \vec{\Phi} + \alpha\Delta\vec{\Phi} + \mathcal{O}(\alpha^2)$. With $e^{i\alpha} = 1 + i\alpha + \mathcal{O}(\alpha^2)$ the given expressions for $\Delta\vec{\Phi}$ and $\Delta\vec{\Phi}^\dagger$ follow.