Sol. 3 SU(N) gauge theory for $N \geq 2$, as given on p. 9-11 of the lecture notes.

(a) Let's try a QED-style charge scaling: $\theta^a(x) \to Q\,\theta^a(x)$, $g \to Q g$.

Demanding that W^a_μ transforms in the same way as before charge scaling, i.e. $W^a_\mu \to W^a_\mu - \frac{1}{g}\partial_\mu \theta^a - f^{abc}\theta^b W^c_\mu$, forces us to also scale $f^{abc} \to \frac{1}{Q} f^{abc}$.

However, in view of $[T^a, T^b] = i f^{abc} T^c$, this corresponds to performing the scaling $T^a \to \frac{1}{Q} T^a$. The net effect of all these scalings is that the gauge and Yang-Mills interactions remain unchanged, as gT^a and gf^{abc} are invariant under the scaling. So, we conclude that the <u>interaction strength of a non-Abelian gauge theory is fixed!</u>

(b) Next we consider the subset of <u>global SU(N) transformations</u> and try to find the corresponding N^2-1 Noether currents (belonging to T^a).

Infinitesimal SU(N) transformations of the various fields:

$$\psi_j(x) \to \psi_j(x) + \theta^a \underbrace{\left[i (T^a)_{jk} \psi_k \right]}_{\equiv \Delta^a \psi_j} \;,\quad \bar{\psi}_j(x) \to \bar{\psi}_j(x) + \theta^a \underbrace{\left[-i \bar{\psi}_k (T^a)_{kj} \right]}_{\equiv \Delta^a \bar{\psi}_j} ,$$

$$W^b_\nu(x) \to W^b_\nu(x) - \underbrace{\cancel{\tfrac{1}{g}\partial_\nu \theta^b}}_{=0 \text{ (global transf.)}} + \theta^a \underbrace{\left[-f^{bac} W^c_\nu(x) \right]}_{\equiv \Delta^a W^b_\nu} .$$

The corresponding Noether currents are (for each independent θ^a):

$$J^{a,\mu}(x) \underset{\mathcal{L}\text{ inv.}}{\overset{G^{a,\mu}=0}{=}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_j)} \Delta^a \psi_j + \Delta^a \bar{\psi}_j \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_j)}} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu W^b_\nu)} \Delta^a W^b_\nu$$

$$= -\bar{\psi}_j \gamma^\mu (T^a)_{jk} \psi_k - W^{b,\mu\nu} f^{abc} W^c_\nu = \underbrace{-\bar{\psi}\gamma^\mu T^a \psi}_{-j^{a,\mu}(x)} \underbrace{- W^{b,\mu\nu} f^{abc} W^c_\nu}_{\text{extra gauge-boson terms}} .$$

The combination $g W^a_\mu(x) J^{a,\mu}(x)$ then reads: $g W^a_\mu(x) J^{a,\mu}(x) =$

$$\underbrace{-g\, j^{a,\mu}(x) W^a_\mu(x)}_{\text{gauge int.}} \underbrace{- g f^{abc} \left(\partial^\mu W^{b,\nu}(x) - \partial^\nu W^{b,\mu}(x) \right) W^a_\mu(x) W^c_\nu(x)}_{2 * \text{triple gauge-boson int.}}$$

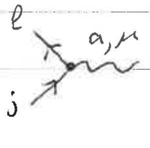
$$\underbrace{+ g^2 f^{abc} f^{bde} W^a_\mu(x) W^c_\nu(x) W^{d,\mu}(x) W^{e,\nu}(x)}_{4 * \text{quartic gauge-boson int.}} .$$

also present in QED case

NOT present in QED case !

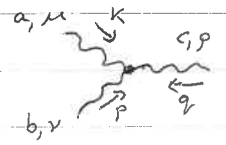
Sol. 4 <u>Feynman rules</u> for the three <u>SU(N) interaction vertices</u>.

<u>Gauge interaction</u>: $-g \bar{\psi}_\ell(x) \gamma^\mu (T^a)_{\ell j} \psi_j(x) W^a_\mu(x) = \mathcal{L}^{(1)}_{int} = -\mathcal{H}^{(1)}_{int}$

$\to$  $= i * \left[-g \gamma^\mu (T^a)_{\ell j} \right] = -i g \gamma^\mu (T^a)_{\ell j}$.

Triple gauge-boson interaction: $g\,[(\partial^{\nu} W^{a}_{\mu}(x)]\,W^{b}_{\nu}(x)\,W^{c}_{\rho}(x)\,f^{abc}\,g^{\rho\mu} = \mathcal{L}^{(2)}_{int}$

$i * (-\text{incoming momentum})^{\nu}$ from $\hat{a}$-term

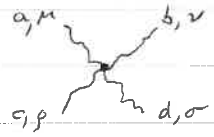
→ 

$= i * \big[-ik^{\nu} g^{\rho\mu} g f^{abc}$ + all other permutations of (a,μ,k), (b,ν,p) and $(c,\rho,q)\big]$

$= -g f^{abc}\big[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}\big]$,

where we have used that f^{abc} is totally antisymmetric to determine the sign of all six different permutations.

Quartic gauge-boson interaction: $-\frac{g^2}{4} f^{abe} f^{cde}\, W^{a}_{\mu}(x) W^{b}_{\nu}(x) W^{c}_{\rho}(x) W^{d}_{\sigma}(x)\, g^{\mu\rho} g^{\nu\sigma} = \mathcal{L}^{(3)}_{int}$

→ 

$= -\frac{ig^2}{4}\big[f^{abe} f^{cde} g^{\mu\rho} g^{\nu\sigma}$ + all other permutations of $\underbrace{(a,\mu)}_{1}$, $\underbrace{(b,\nu)}_{2}$, $\overbrace{(c,\rho)}^{3}$ and $\overbrace{(d,\sigma)}^{4}\big]$

$= -ig^2\big[f^{abe} f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace} f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$
$+ f^{ade} f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]$,

since all terms receive four contributions (e.g. $1\leftrightarrow 2, 3\leftrightarrow 4$ and $1\leftrightarrow 3, 2\leftrightarrow 4$ and $1\leftrightarrow 4, 2\leftrightarrow 3$ yield the same $f^{abe} f^{cde} g^{\mu\rho} g^{\nu\sigma}$ contribution).

QFT-style way to determine Feynman rules in coordinate space:

consider for instance the fully connected contributions to the Green's function

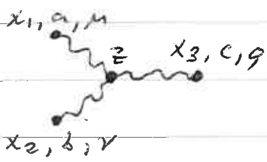
$\langle 0|\,T\big(\hat{W}^{a}_{I_\mu}(x_1)\,\hat{W}^{b}_{I_\nu}(x_2)\,\hat{W}^{c}_{I_\rho}(x_3)\,\big[ig f^{a_1 a_2 a_3} g^{\alpha_3\alpha_1}\int d^4z\,(\partial^{\alpha_2}\hat{W}^{a_1}_{I_{\alpha_1}}(z))\,\hat{W}^{a_2}_{I_{\alpha_2}}(z)\,\hat{W}^{a_3}_{I_{\alpha_3}}(z)\big]\big)|0\rangle$

From the 1st-order $i\int d^4z\,\hat{\mathcal{L}}_{int_I}(z)$ part of $\hat{U}(\infty,-\infty)$

in order to obtain the TGC Feynman rule

$\Rightarrow\ ig f^{a_1 a_2 a_3} g^{\alpha_3\alpha_1}\int d^4z\,\langle 0|\,\hat{W}^{a}_{I_\mu}(x_1)\,\hat{W}^{b}_{I_\nu}(x_2)\,\hat{W}^{c}_{I_\rho}(x_3)\,(\partial^{\alpha_2}\hat{W}^{a_1}_{I_{\alpha_1}}(z))\,\hat{W}^{a_2}_{I_{\alpha_2}}(z)\,\hat{W}^{a_3}_{I_{\alpha_3}}(z)|0\rangle$

+ five other permutations contracting the first three fields with the other three fields from the interaction term, in order to obtain all fully connected contributions

= 

the translation to the momentum-space Feynman rules is given above