Sol. 5 Free massive spin-1 theory (Proca theory): $\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$,
with A_μ a real gauge field, $m \neq 0$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Equation of motion: $\partial_\mu \frac{\partial \mathcal{L}_{Proca}}{\partial(\partial_\mu A_\nu)} = \underbrace{-\partial_\mu F^{\mu\nu}}_{-\Box A^\nu + \partial^\nu(\partial.A)} = \frac{\partial \mathcal{L}_{Proca}}{\partial A_\nu} = m^2 A^\nu$

$\Rightarrow$ Proca eqn. $\boxed{(\Box + m^2)A^\nu - \partial^\nu(\partial.A) = 0}$ for $\nu = 0, \ldots, 3$ (1).

(b) $\partial_\nu * (1) = (\Box + m^2)\partial.A - \Box(\partial.A) = m^2 \partial.A = 0 \xrightarrow{m\neq 0}$ Lorenz condition $\boxed{\partial.A = 0}$.

(c) (1) $\xrightarrow{\partial.A=0}$ $(\Box + m^2)A^\nu = 0$, i.e. the Klein-Gordon eqn. holds for $A^0, \ldots, A^3$.
As such, the solutions to the Proca eqn. can be decomposed into plane waves (being solutions to the KG eqn.).

(d) Quantum-field solution to the Proca eqn. (Proca field):

$$\hat{A}_\mu(x) = \sum_{\lambda} \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\epsilon^\lambda_\mu(p)\hat{a}(\vec{p},\lambda)e^{-ip.x} + \epsilon^{\lambda *}_\mu(p)\hat{a}^\dagger(\vec{p},\lambda)e^{+ip.x}\right)\Big|_{p^0 = E_{\vec{p}}},$$

(pol.) on-shell condition $p^2 = m^2$

with $p^\mu = (p^0, \vec{p}) = (\sqrt{\vec{p}^2 + m^2}, \vec{p})$ and $\epsilon^\lambda_\mu(p)$ the polarization vector (spin d.o.f.).

Lorenz condition: $0 = \partial^\mu \hat{A}_\mu(x) = \sum_\lambda \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(-ip.\epsilon^\lambda(p)\,\hat{a}(\vec{p},\lambda)e^{-ip.x} + h.c.\right)\Big|_{p^0 = E_{\vec{p}}}$.

Since the plane waves form a basis, this implies that $\boxed{p^\mu \epsilon^\lambda_\mu(p) = 0}$.

(e) Propagator: $\langle 0|T(\hat{A}_\mu(x)\hat{A}^\dagger_\nu(y))|0\rangle = \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip.(x-y)}}{p^2 - m^2 + i\epsilon}}_{\text{scalar part, with on-shell pole @ } p^2 = m^2} \underbrace{\left(\sum_\lambda \epsilon^\lambda_\mu(p)\epsilon^{\lambda *}_\nu(p)\Big|_{p^0 = E_{\vec{p}}}\right)}_{\equiv R_{\mu\nu}\text{, polarization part}}$

Apply Lorentz covariance to $R_{\mu\nu}$: which objects can carry the indices μ and ν?
$\rightarrow$ Since $p.\epsilon^\lambda(p) = 0$, $R_{\mu\nu}$ can only depend on $g_{\mu\nu}$ and p_μ, p_ν
[as no other vectors feature in the polarization vectors].
Hence, $R_{\mu\nu}(p) \propto c_1 g_{\mu\nu} + c_2 p_\mu p_\nu$.

Mass dimensions: $[A_\mu] = 1$, since $[m^2 A_\mu A^\mu] = 2 + 2[A_\mu] = [\mathcal{L}] = 4$
$\Rightarrow$ [propagator] $= 2$ and $[R_{\mu\nu}(p)] = 0$.
(like in scalar case) $\Rightarrow$ $[c_1] = 0$, $[c_2] = -2$

Lorenz condition: $p^\mu R_{\mu\nu}(p) = R_{\mu\nu}(p)p^\nu \overset{(d)}{=} 0 \xrightarrow{p^2 = m^2} c_1 p_\nu + c_2 m^2 p_\nu = 0$, i.e. $c_2 = \frac{-c_1}{m^2}$
and hence $R_{\mu\nu}(p) \propto c_1(g_{\mu\nu} - p_\mu p_\nu/m^2)$.

(f) Proca eqn. (1) $= \left[(\Box + m^2)g^{\rho\nu} - \partial^\nu\partial^\rho\right]A_\rho \equiv D^{\nu\rho}_{Proc}A_\rho = 0$.
Momentum space: $D^{\mu\nu}_{Proc} \rightarrow D^{\mu\nu}_{Proc}(p) = p^\mu p^\nu - [p^2 - m^2]g^{\mu\nu}$.

(g) Full propagator in momentum space according to (e): $P_{\mu\nu} = c_3(p^2)g_{\mu\nu} + c_4(p^2)p_\mu p_\nu$
Demand that the propagator is the inverse of the eqn. of motion:
$\underbrace{D^{\mu\nu}_{Proc}(p)P_{\nu\lambda}(p)}_{} \equiv i\delta^\mu{}_\lambda = i g^\mu{}_\lambda$.
$\Rightarrow -g^\mu{}_\lambda(p^2 - m^2)c_3(p^2) + p^\mu p_\lambda[c_3(p^2) \overbrace{- c_4(p^2)(p^2 - m^2) + c_4(p^2)p^2}^{+m^2 c_4(p^2)}]$

Hence, $c_3(p^2) = \frac{-i}{p^2 - m^2} = -m^2 c_4(p^2) \Rightarrow \boxed{P_{\mu\nu}(p) = \frac{i}{p^2 - m^2}(-g_{\mu\nu} + p_\mu p_\nu/m^2)}$.

(h) Large-momentum behaviour for propagators inside loop diagrams: consider all components of p^μ to be of order $\Lambda \gg m$.

$\Rightarrow \dfrac{-ig_{\mu\nu}}{p^2-m^2} \propto 1/\Lambda^2$, similar to theories with a scalar force mediator;

$\dfrac{ip_\mu p_\nu/m^2}{p^2-m^2} \propto \Lambda^0$, these terms are potentially much larger than the "scalar ones" unless they cancel somehow [e.g. through Ward identities ... which, however, are invalidated by the absence of U(1) gauge invariance due to the gauge-boson mass term].

(i) Consider "massive QED": $\mathcal{L} = i\bar\psi\gamma^\mu D_\mu\psi - m_f\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$,

$D_\mu\psi = (\partial_\mu + igA_\mu)\psi \Rightarrow \mathcal{L}_{int} = -g\bar\psi\gamma^\mu\psi A_\mu$.

Since $[g]=0$, this theory is expected to be renormalizable! $[g]=0$

unless ...

Arbitrary amputated loop diagram: N_F = # external fermions, L = # loops,
N_B = # external bosons, V = # $\bar\psi\psi A$ vertices,
P_F = # fermion propagators,
P_B = # boson propagators.

Identities: *) fermions $\rightarrow 2V = N_F + 2P_F$; bosons $\rightarrow V = N_B + 2P_B$

$\Rightarrow P_F = V - N_F/2,\ P_B = V/2 - N_B/2$;

*) # undetermined loop momenta $= L = P - V + 1 = P_F + P_B - V + 1$ as usual; $\int d^4 l$

*) naive power counting: each loop momentum $\rightarrow \Lambda^4$, each fermion propagator $\left(\dfrac{\not{p}+m_f}{p^2-m_f^2+i\epsilon}\right) \rightarrow \Lambda^{-1}$, each boson propagator $\xrightarrow{(h)} \Lambda^0$, each vertex $\rightarrow \Lambda^0$.

$\Rightarrow$ superficial degree of divergence $D = 4L - P_F = 3P_F + 4P_B - 4V + 4$
$= 4 + V - \frac{3}{2}N_F - 2N_B$

positive coeff. $\Rightarrow$ non-renormalizable theory; at sufficiently high loop order each amplitude becomes divergent !