Sol. 6 Standard Model : $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge theory.
Let's consider some sample covariant derivatives.

6 Dirac spinors

- Left-handed quark isospin doublet $Q_L \equiv \begin{pmatrix} \psi_{u_L} \\ \psi_{d_L} \end{pmatrix}$.

with $\psi_{u_L} = \begin{pmatrix} (\psi_{u_L})_R \\ (\psi_{u_L})_G \\ (\psi_{u_L})_B \end{pmatrix}$ and $\psi_{d_L} = \begin{pmatrix} (\psi_{d_L})_R \\ (\psi_{d_L})_G \\ (\psi_{d_L})_B \end{pmatrix}$ colour triplets

$\hookrightarrow$ covariant derivative $D_\mu(Q_L) = \partial_\mu + \frac{ig_s}{2}\lambda^a G_\mu^a + \frac{ig}{2}\sigma^i W_\mu^i + \frac{ig'}{2} Y(Q_L) B_\mu$,
with $a = 1, \dots, 8$ for $SU(3)_c$, $i = 1,2,3$ for $SU(2)_L$ and hypercharge $Y(Q_L) = 1/3$.

$\lambda^a \rightarrow$ *) The Gell-Mann λ-matrices act as 3×3 matrices on each of the fundamental colour triplets ψ_{u_L} and ψ_{d_L}, but they don't mix the upper and lower half of the isospin doublet !

$\sigma^i \rightarrow$ *) The Pauli spin matrices act as 2×2 matrices on the isospin doublet without mixing the three components (labeled by R, G and B) of the colour triplets that make up the upper and lower half of the isospin doublet !

1 Dirac spinor

- Right-handed electron isospin singlet ψ_{e_R}
$\hookrightarrow$ covariant derivative $D_\mu(e_R) = \partial_\mu + \frac{ig'}{2} Y(e_R) B_\mu$,
with hypercharge $Y(e_R) = -2$.

Sol. 7 Consider the complex Klein-Gordon theory with mass m:
$\mathcal{L}(x) = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi$, in terms of the complex scalar field $\phi \equiv \phi(x)$.

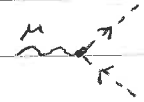
(a) Replace ∂_μ by $D_\mu = \partial_\mu + igA_\mu$, with $A_\mu \equiv A_\mu(x)$ a real gauge field.
The resulting Lagrangian is invariant under the local gauge transformation $\phi(x) \to \phi'(x) = e^{i\vartheta(x)}\phi(x)$, $\phi^*(x) \to \phi'^*(x) = e^{-i\vartheta(x)}\phi^*(x)$
provided that we demand $A_\mu(x)$ to transform according to
$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\vartheta(x)$.

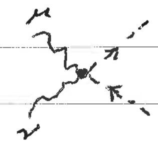
Proof:
$$\begin{aligned} D_\mu\phi(x) \to D'_\mu\phi'(x) &= \left[\partial_\mu + ig\left(A_\mu(x) - \tfrac{1}{g}\partial_\mu\vartheta(x)\right)\right] e^{i\vartheta(x)}\phi(x) \\ &= e^{i\vartheta(x)}\left[\partial_\mu + \cancel{(i\partial_\mu\vartheta(x))} + igA_\mu(x) - \cancel{(i\partial_\mu\vartheta(x))}\right]\phi(x) \\ &= e^{i\vartheta(x)} D_\mu\phi(x). \end{aligned}$$

Consequently $(D_\mu\phi)^*(D^\mu\phi)$ is gauge invariant, just like $-m^2\phi^*\phi$ trivially is.

(b) $(D_\mu\phi)^*(D^\mu\phi) = (\partial_\mu\phi^* - igA_\mu\phi^*)(\partial^\mu\phi + igA^\mu\phi)$

$$= (\partial_\mu\phi)^*(\partial^\mu\phi) + \underbrace{ig\left[(\partial_\mu\phi)^*\phi A^\mu - \phi^*(\partial^\mu\phi)A_\mu\right]} + \underbrace{g^2\phi^*\phi A_\mu A^\mu}$$

interaction term involving one gauge boson and two scalar particles :  ← (arrow needed since $\phi \neq \phi^*$)

interaction term involving two gauge bosons and scalar particles : 

(c) A mass term for the A_μ-field would look like $+\frac{1}{2}m_A^2 A_\mu A^\mu$, which is not invariant under the gauge transformation $A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \vartheta(x)$

$\Rightarrow$ adding such a mass term by hand results in a broken gauge theory.

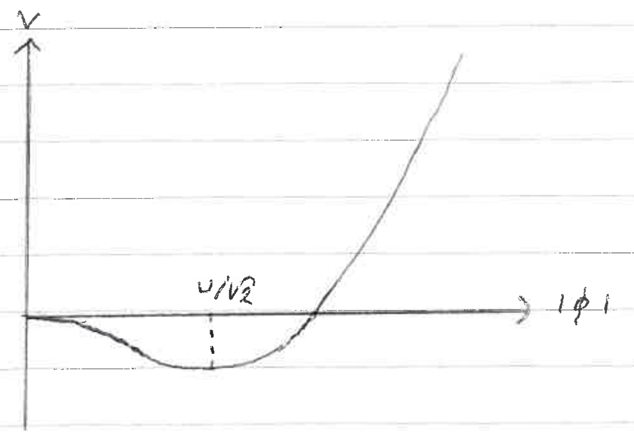
New set-up : $\mathcal{L}(x) = (\partial_\mu\phi)^*(\partial^\mu\phi) - V(\phi)$,
with $V(\phi) = c_2|\phi|^2 + c_4|\phi|^4$ and $|\phi| = \sqrt{\phi^*\phi}$.

(d) + (e) Local maximum at $|\phi| = 0$, global (ring of) minima at $|\phi| > 0$:

*) $V(\phi)$ should be bounded from below $\Rightarrow$ $c_4 > 0$;

*) $\dfrac{\partial V}{\partial|\phi|} = 2c_2|\phi| + 4c_4|\phi|^3 = 2c_2|\phi|\left(1 + \dfrac{2c_4}{c_2}|\phi|^2\right)$

$\Rightarrow$ global minima at $|\phi| > 0$ if $2c_4/c_2 < 0$, i.e. $c_2 < 0$.



Location of minima :
$$|\phi_{min}|^2 = -\frac{c_2}{2c_4} = v^2/2 .$$

Freedom to choose the location of the minimum for ϕ :
we take it to be $\phi_{min} = v/\sqrt{2} \in \mathbb{R}$ and reparametrize ϕ according to $\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))\, e^{i w(x)/v}$ ($h(x), w(x) \in \mathbb{R}$ scalar fields).

(f) The modes (fields) $h(x)$ and $w(x)$ describe the fluctuations w.r.t. the chosen minimum of the potential:

original $U(1)$ symmetry, spontaneously broken by choosing the minimum

$w(x)$ = mode connecting the ring of minima (representing the phase),
$h(x)$ = mode perpendicular to the ring (in the real direction),

$$\phi(x) = \frac{1}{\sqrt{2}}(v+h(x))\,[\underbrace{\cos(w(x)/v)}_{\text{real part}} + \underbrace{i\sin(w(x)/v)}_{\text{imaginary part}}].$$

(g) + (h)

$$\mathcal{L}(x) = \tfrac{1}{2}[\partial_\mu h - i(v+h)\frac{\partial_\mu w}{v}]\,e^{-iw/v}\,[\partial^\mu h + i(v+h)\frac{\partial^\mu w}{v}]\,e^{iw/v}$$

$$- \frac{c_2}{2}(v+h)^2 - \frac{c_4}{4}(v+h)^4$$

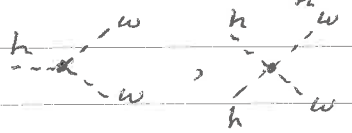
$$= \overbrace{\tfrac{1}{2}(\partial_\mu h)(\partial^\mu h)}^{\text{kinetic term for } h} + \tfrac{1}{4}c_4 v^4 + 0.h \overbrace{- c_4 v^2 h^2}^{\text{mass term for } h \,\equiv\, -\frac{1}{2}m_h^2 h^2} \underbrace{- c_4 v h^3 - \tfrac{1}{4}c_4 h^4}$$

interactions involving h :

$$+ \tfrac{1}{2}(1+h/v)^2(\partial_\mu w)(\partial^\mu w)$$

kinetic term for w

interactions involving both h and w :



*) The boson described by the h-field (<u>Higgs boson</u>) has acquired a mass $m_h = \sqrt{2c_4 v^2} = \sqrt{-2c_2}$;

*) The w-field corresponds to a massless boson (<u>Goldstone boson</u>).

(i)
$$(D_\mu\phi)^*(D^\mu\phi) = \tfrac{1}{2}[\partial_\mu h - i(v+h)(gA_\mu + \partial_\mu w/v)]\,e^{-iw/v}\,[\partial^\mu h + i(v+h)(gA^\mu + \partial^\mu w/v)]\,e^{iw/v}$$
$$= \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{2}(1+h/v)^2[(\partial_\mu w)(\partial^\mu w) + g^2v^2 A_\mu A^\mu + 2vgA^\mu\partial_\mu w]$$

(j) The term $vgA^\mu\partial_\mu w$ is the odd one out, since it is not a kinetic term nor an obvious mass term. It represents <u>mixing between the A_μ and $\partial_\mu w$ fields</u> in the (non-interaction) part of the Lagrangian from which the lowest-order particle content of the theory, i.e. the mass eigenstates, should be derived.

Problem: we have to get rid of this **mixing** term and diagonalize the non-interaction part of this theory!

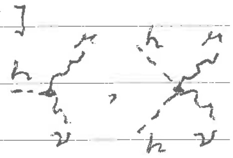
<u>Solution</u>: the true lowest-order particle content is revealed by choosing a specific gauge to work in, the so-called <u>unitary gauge</u>:

$$\phi^{(u)}(x) = e^{-i\omega(x)/v}\,\phi(x) = \frac{1}{\sqrt{2}}\,(v + h(x)),$$
$$A_\mu^{(u)}(x) = A_\mu(x) + \frac{1}{gv}\,\partial_\mu \omega(x).$$

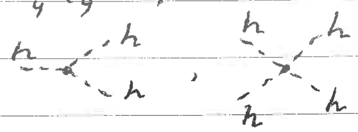
(k) In that case $(D_\mu \phi^{(u)})^* (D^\mu \phi^{(u)}) = \frac{1}{2}[\partial_\mu h - igA_\mu^{(u)}(v+h)][\partial^\mu h + igA^{(u),\mu}(v+h)]$
$= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}A_\mu^{(u)}A^{(u),\mu}(g^2v^2 + 2g^2vh + g^2h^2).$

(l) Combining parts (g) and (k) we get

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) - c_4 v^2 h^2}_{\text{kinetic and mass term for } h} + \underbrace{\tfrac{1}{2}A_\mu^{(u)}A^{(u),\mu}[g^2v^2}_{\text{mass term for } A_\mu^{(u)} \,\equiv +\frac{1}{2}m_A^2 A_\mu^{(u)}A^{(u),\mu}} + \underbrace{2g^2vh + g^2h^2]}_{\text{interactions}}$$

interactions: 

$$\underbrace{-c_4 v h^3 - \tfrac{1}{4}c_4 h^4}_{\text{interactions}} + \tfrac{1}{4}c_4 v^4,$$

interactions: 

Features: only the h-modes have remained, the ω-modes have vanished from the expressions ! The field $A_\mu^{(u)}$ corresponds to a mass eigenstate of the theory with non-zero mass $m_A = gv$!

(m) The gauge boson has become massive by selecting the minimum of the potential (from a degenerate set) that should be used as "ground state" and around which the quantum fluctuations should be considered. This breaks the original U(1) gauge symmetry, which is reflected by the ring of minima from which the ground state could have been selected.

This way of breaking the symmetry without breaking it is called spontaneous symmetry breaking !

A massless spin-1 gauge boson has two polarization d.o.f. [helicity ±1], whereas a massive gauge boson has an extra longitudinal d.o.f. corresponding to helicity 0. This extra d.o.f. has been supplied by the Goldstone mode ω, as the unitary-gauge expression has revealed (by "gauging away" ω) !

This is called the Higgs mechanism !