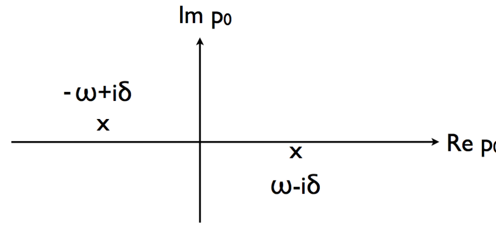
Solution 5:

Free complex Klein-Gordon field: $\mathcal{L} = (\partial_\mu \phi^\star)(\partial^\mu \phi) - m^2 \phi^\star \phi$.

(a) Where are the poles of the Feynman propagator?
Note that ϵ has to be understood as $\epsilon \to 0^+$:



$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{\left(p_0 - \sqrt{\vec{p}^{\,2} + m^2 - i\epsilon}\,\right)\left(p_0 + \sqrt{\vec{p}^{\,2} + m^2 - i\epsilon}\,\right)},$$

where

$$\sqrt{\vec{p}^{\,2}+m^2 - i\epsilon} = \sqrt{\vec{p}^{\,2}+m^2} - \frac{i\epsilon}{2\sqrt{\vec{p}^{\,2}+m^2}} + \mathcal{O}(\epsilon^2) = \omega_{\vec{p}} - \frac{i\epsilon}{2\omega_{\vec{p}}} + \mathcal{O}(\epsilon^2) \equiv \omega_{\vec{p}} - i\delta + \mathcal{O}(\delta^2).$$

With $\epsilon \to 0^+$ also $\delta = \frac{\epsilon}{2\omega_{\vec{p}}} \to 0^+$, and the poles in the complex p_0-plane coincide with the prescription on page 26 of the lecture notes, which yields the Feynman propagator after the integration is performed.

(b) The field operator $\hat{\phi}(x)$ contains the operators $\hat{a}_{\vec{p}}$ and $\hat{b}^\dagger_{\vec{p}}$: $\hat{\phi}(x) = \cdots \hat{a}_{\vec{p}} + \cdots \hat{b}^\dagger_{\vec{p}}$ such that

$$\langle 0| T\big(\hat{\phi}(x)\hat{\phi}(y)\big)|0\rangle = \langle 0| \ldots \hat{a}_{\vec{p}}\hat{a}_{\vec{q}} + \ldots \hat{a}_{\vec{p}}\hat{b}^\dagger_{\vec{q}} + \ldots \hat{b}^\dagger_{\vec{p}}\hat{a}_{\vec{q}} + \ldots \hat{b}^\dagger_{\vec{p}}\hat{b}^\dagger_{\vec{q}}|0\rangle .$$

The first, third and fourth terms vanish directly by either acting with $\hat{a}_{\vec{p}}$ to the right or with $\hat{b}^\dagger_{\vec{p}}$ to the left on the vacuum. In the second term one has first to commute the operators, which does not give any extra term since $[\hat{a}_{\vec{p}}, \hat{b}^\dagger_{\vec{p}}] = 0$. The fact that this amplitude vanishes can also be understood physics-wise. First an antiparticle is being created out of the vacuum at spacetime point y (or x), whereas subsequently a particle is being annihilated at spacetime point x (or y). Obviously this cannot correspond to the propagation of an actual (anti)particle.
For $\langle 0| T\big(\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)\big)|0\rangle$ the same arguments apply, the only difference being the appearence of the operators $\hat{a}^\dagger_{\vec{p}}$ and $\hat{b}_{\vec{p}}$. This just interchanges the role of particles and antiparticles.

(c) Using that $\big[\hat{H}, \hat{a}_{\vec{p}}\big]e^{-ip\cdot x} = -\omega_{\vec{p}}\,\hat{a}_{\vec{p}}\,e^{-ip\cdot x} = -i\partial_0\big(\hat{a}_{\vec{p}}\,e^{-ip\cdot x}\big)$ and $\big[\hat{H}, \hat{a}^\dagger_{\vec{p}}\big]e^{ip\cdot x} = \omega_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x} = -i\partial_0\big(\hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x}\big)$, we can write an infinitesimal time translation of $\hat{\phi}(x)$ as being generated by $\hat{H}$:

$$\hat{\phi}(x) + \Delta t\, \partial_0 \hat{\phi}(x) = \hat{\phi}(x) + i\Delta t\big[\hat{H}, \hat{\phi}(x)\big] \approx e^{i\hat{H}\Delta t}\,\hat{\phi}(x)\,e^{-i\hat{H}\Delta t} \qquad (\Delta t \in \mathbb{R} \text{ infinitesimal}).$$

Solution 6:

Consider the time-ordered exponential of the operator $\hat{A}(t)$ for $\tau \le t$:

$$\hat{E}(t,\tau) = \hat{1} + \int_\tau^t dt_1\, \hat{A}(t_1) + \int_\tau^t dt_1\, \hat{A}(t_1) \int_\tau^{t_1} dt_2\, \hat{A}(t_2) + \ldots$$

(a) $\hat{E}(t,\tau)$ satisfies the boundary condition $\hat{E}(\tau,\tau) = \hat{1}$ because $\int_\tau^\tau dt_1 \hat{A}(t_1) = 0$ (zero integration measure). As $\frac{\partial}{\partial t}\int_\tau^t dt_1 \hat{A}(t_1) = \hat{A}(t)$ (differentiating the upper limit of an integral gives the integrand evaluated at the upper limit), $\hat{E}(t,\tau)$ fulfills the linear differential equation

$$\frac{\partial}{\partial t}\,\hat{E}(t,\tau) = 0 + \hat{A}(t) + \hat{A}(t)\int_\tau^t dt_2\,\hat{A}(t_2) + \hat{A}(t)\int_\tau^t dt_2\,\hat{A}(t_2)\int_\tau^{t_2} dt_3\,\hat{A}(t_3) + \ldots = \hat{A}(t)\hat{E}(t,\tau)\,.$$

(b) To Prove: $\hat{E}(t,\tau) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\int_\tau^t dt_1\,\ldots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_n)\big)\,.$

The decisive step in the proof:
$\frac{\partial}{\partial t}\int_\tau^t dt_1\ldots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_n)\big)$ leads to n terms, such that the ith term has $i-1$ terms to the left and $n-i$ terms to the right of the operator $\hat{A}(t)$. Now, t is the latest time, and the time ordering operator implies that the operator $\hat{A}(t)$ has to be pulled to the leftmost position. The above derivative results in $n\hat{A}(t)\int_\tau^t dt_1\ldots\int_\tau^t dt_{n-1}\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_{n-1})\big)$ and therefore one has

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{1}{n!}\int_\tau^t dt_1\,\ldots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_n)\big)\\
&= \hat{A}(t)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int_\tau^t dt_1\,\ldots\int_\tau^t dt_{n-1}\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_{n-1})\big)\\
&= \hat{A}(t)\sum_{n=0}^{\infty}\frac{1}{n!}\int_\tau^t dt_1\,\ldots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_n)\big)\,.
\end{aligned}$$

We have just seen that the time-ordered operator

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int_\tau^t dt_1\,\ldots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\ldots\hat{A}(t_n)\big)$$

satisfies the same linear differential equation as $\hat{E}(t,\tau)$. Since this time-ordered operator also satisfies the same boundary condition as $\hat{E}(t,\tau)$, i.e. yielding $\hat{1}$ at $t=\tau$, it must indeed be identical to $\hat{E}(t,\tau)$.

(c) If the operators $\hat{A}(t)$ commute for all times (the operators are then like ordinary numbers) the T-ordering is clearly not needed, because all orderings are then equivalent:

$$\hat{E}(t,\tau) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_\tau^t dt'\,\hat{A}(t')\right)^n\,.$$

One obtains then the usual exponential function

$$\hat{E}(t,\tau) \;=\; e^{\int_\tau^t dt'\,\hat{A}(t')}\,,$$

with the calculational rules as known from basic calculus.