Solution 9:

(a) In order to calculate decay and scattering amplitudes $\langle f|\hat{S}|i\rangle = i\mathcal{M}(2\pi)^4\delta^{(4)}(p_i - p_f)$ one needs in addition to the Feynman rules derived in exercise 7 also the contractions of field operators with external lines.

$\boxed{\text{In coordinate space}}$: the field operator $\hat{\psi}_{\mathrm{I}}(x)$ can be contracted with an initial ψ-particle state on the right providing a factor $e^{-ik_\psi\cdot x}$, since

$$\overline{\hat{\psi}_{\mathrm{I}}(x)|\vec{k}_\psi}\rangle_0 = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{q}}}}\, e^{-i\omega_{\vec{q}}x^0 + i\vec{q}\cdot\vec{x}}\, \hat{b}_{\vec{q}}\sqrt{2\omega_{\vec{k}_\psi}}\,\hat{b}^\dagger_{\vec{k}_\psi}|0\rangle = e^{-ik_\psi\cdot x}|0\rangle\,.$$

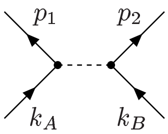
Analogously, a contraction ${}_0\langle\overline{\vec{p}_{\bar\psi}|\hat{\psi}_{\mathrm{I}}(x)}$ of the field operator $\hat{\psi}_{\mathrm{I}}(x)$ with a final $\bar\psi$-particle state on the left provides a factor $e^{ip_{\bar\psi}\cdot x}$, a contraction $\overline{\hat{\psi}^\dagger_{\mathrm{I}}(x)|\vec{k}_{\bar\psi}}\rangle_0$ of the field operator $\hat{\psi}^\dagger_{\mathrm{I}}(x)$ with an initial $\bar\psi$-particle state on the right provides a factor $e^{-ik_{\bar\psi}\cdot x}$, and a contraction ${}_0\langle\overline{\vec{p}_\psi|\hat{\psi}^\dagger_{\mathrm{I}}(x)}$ of the field operator $\hat{\psi}^\dagger_{\mathrm{I}}(x)$ with a final ψ-particle state on the left provides a factor $e^{ip_\psi\cdot x}$.

To mark this in Feynman diagrams an arrow will be drawn on $\psi/\bar\psi$-lines, representing the direction of particle number flow: ψ-particles flow along the arrow, $\bar\psi$-particles = ψ-antiparticles against the arrow.

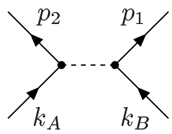
$\boxed{\text{In momentum space}}$: these extra Feynman rules all provide a trivial factor 1.

(b) The lowest-order amplitude for the decay process $\phi(k_A) \to \psi(p_1)\bar\psi(p_2)$ is given by the elementary vertex, hence $i\mathcal{M}^{LO}(\phi \to \psi\bar\psi) = -ig$.

(c) In all three $2 \to 2$ scattering cases there are two Feynman diagrams possible at lowest order. With the help of energy-momentum conservation, the lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2)\big)$ reads

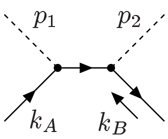


$+$

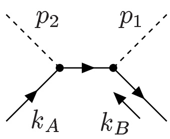


$$= -ig^2\left(\frac{1}{(k_A - p_1)^2 - m^2 + i\epsilon} + \frac{1}{(k_A - p_2)^2 - m^2 + i\epsilon}\right).$$

The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\bar\psi(k_B) \to \phi(p_1)\phi(p_2)\big)$ is given by



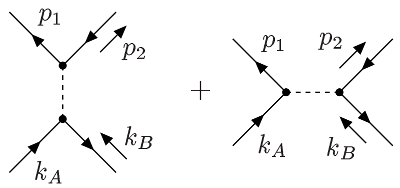
$+$



$$= -ig^2\left(\frac{1}{(k_A - p_1)^2 - M^2 + i\epsilon} + \frac{1}{(k_A - p_2)^2 - M^2 + i\epsilon}\right).$$

Note that the incoming momentum k_B of the antiparticle is defined in the opposite direction of the arrow! This is typical for the arrow convention for antiparticles.

The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2)\big)$ is given by



$$= -\,ig^2\left(\frac{1}{(k_A+k_B)^2 - m^2 + i\epsilon} + \frac{1}{(k_A-p_1)^2 - m^2 + i\epsilon}\right).$$

This amplitude can be obtained from the first one by means of crossing, i.e. changing the momenta according to $k_B\,, p_2 \to -p_2\,, -k_B$ in order to switch from particles to antiparticles in the opposite reaction state.

(d) The lowest-order scattering amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \phi(p_1)\phi(p_2)\big)$ vanishes, since an initial state with particle number 2 cannot scatter into a final state with particle number 0 (cf. exercise 8). In the Feynman diagrams this manifests itself by the absence of a consistent (continuous) flow for the particle number in this particular process.

Solution 11:

(a) Consider the decay in the rest frame of the decaying particle: $k_A^\mu = (m,\vec{0})\,,\ p_1^\mu = (E_1,\vec{p})$ and $p_2^\mu = (E_2,-\vec{p})$. First, note that $E_1 = E_2 = \sqrt{M^2+\vec{p}^{\,2}} = m/2$ and therefore $|\vec{p}| = p = \frac{1}{2}\sqrt{m^2-4M^2} \geq 0$, which implies the condition $m \geq 2M$.

According to exercise 9(b) the matrix element is $i\mathcal{M}^{LO}(\phi \to \psi\bar{\psi}) = -\,ig$. Employing page 64 of the lecture notes with E_{CM} replaced by m gives the decay width

$$\Gamma_{\phi\to\bar{\psi}\psi} = \frac{1}{2m}\,g^2\int d\Pi_2 \overset{p.\,64}{=\!=} \frac{g^2}{2m}\,\frac{p}{16\pi^2 m}\int d\Omega = \frac{g^2}{16\pi m}\sqrt{1-4M^2/m^2}\,,$$

where $\int d\Omega = 4\pi$ has been used.

(b) Consider the lowest-order scattering process $\psi(k_A)\bar{\psi}(k_B) \to \phi(p_1)\phi(p_2)$ in the CM frame:

$$k_A^\mu = (E_A,0,0,k)\,,\quad k_B^\mu = (E_B,0,0,-k)\,,\quad p_1^\mu = (E_1,\vec{p}) \ \text{ and } \ p_2^\mu = (E_2,-\vec{p})\,,$$

$$\text{with}\quad E_A = E_B = E_1 = E_2 = \frac{1}{2}\,E_{CM} = \frac{1}{2}\sqrt{s}\,.$$

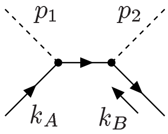
This implies:

$$|\vec{k}| = k = \frac{1}{2}\sqrt{s-4M^2} \geq 0 \quad \Rightarrow \quad E_{CM} = \sqrt{s} \geq 2M\,,$$

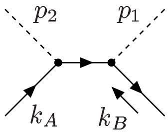
$$|\vec{p}| = p = \frac{1}{2}\sqrt{s-4m^2} \geq 0 \quad \Rightarrow \quad E_{CM} = \sqrt{s} \geq 2m\,,$$

$$\text{and}\quad \vec{k}_A\cdot\vec{p} = kp\cos\theta = \frac{1}{4}\sqrt{s-4M^2}\,\sqrt{s-4m^2}\,\cos\theta\,.$$

According to exercise 9(c) the matrix element $i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \phi(p_1)\phi(p_2)\big)$ reads



$+$



$$= \; -ig^2\left(\frac{1}{(k_A-p_1)^2-M^2}+\frac{1}{(k_A-p_2)^2-M^2}\right),$$

where the diagram on the left is the t-channel diagram and the one on the right the u-channel. Bearing in mind that $t=(k_A-p_1)^2 = m^2+M^2-\frac{s}{2}\Big(1-\sqrt{1-4M^2/s}\,\sqrt{1-4m^2/s}\,\cos\theta\Big)$ and $u=(k_A-p_2)^2 = m^2+M^2-\frac{s}{2}\Big(1+\sqrt{1-4M^2/s}\,\sqrt{1-4m^2/s}\,\cos\theta\Big)$, the matrix element and therefore also the differential cross section are symmetric under $\cos\theta \to -\cos\theta = \cos(\pi-\theta)$. Furthermore, note that the usual infinitesimal imaginary parts have been skipped in the above-given propagators. Since $t\,,u < M^2$, these propagators can never become on-shell and as such the infinitesimal imaginary parts are obsolete.

The lowest order differential cross section then reads

$$\frac{d\sigma^{LO}}{d\Omega} \overset{p.\,65}{=\!=} \frac{\sqrt{1-4m^2/s}}{64\pi^2 s\sqrt{1-4M^2/s}}\left|\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B)\to\phi(p_1)\phi(p_2)\big)\right|^2\Theta(\sqrt{s}-2M)\,\Theta(\sqrt{s}-2m)\,.$$

(c) Consider the lowest-order scattering process $\psi(k_A)\bar{\psi}(k_B)\to\psi(p_1)\bar{\psi}(p_2)$ in the CM frame:

$$k_A^\mu = (E_A,0,0,k)\,,\quad k_B^\mu=(E_B,0,0,-k)\,,\quad p_1^\mu=(E_1,\vec{p})\ \text{ and }\ p_2^\mu=(E_2,-\vec{p})\,,$$
$$\text{with}\quad E_A=E_B=E_1=E_2=\frac{1}{2}\,E_{CM}=\frac{1}{2}\sqrt{s}\,.$$

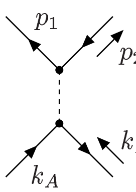
This implies:

$$|\vec{k}| = k = |\vec{p}| = p = \frac{1}{2}\sqrt{s-4M^2}\,\geq\,0 \quad\Rightarrow\quad E_{CM}=\sqrt{s}\,\geq\,2M\,,$$
$$\text{and}\quad \vec{k}_A\cdot\vec{p} = kp\cos\theta = \frac{1}{4}\,(s-4M^2)\cos\theta\,.$$

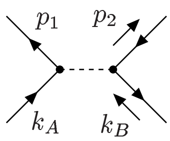
Therefore we obtain: $s\,\geq\,4M^2$, $t=(k_A-p_1)^2 = -\frac{s}{2}\,(1-4M^2/s)(1-\cos\theta)\,\leq\,0$

and $u=(k_A-p_2)^2 = -\frac{s}{2}\,(1-4M^2/s)(1+\cos\theta)\,\leq\,0\,.$

According to exercise 9(c) the matrix element $i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B)\to\psi(p_1)\bar{\psi}(p_2)\big)$ reads



$+$



$$= \; -ig^2\left(\frac{1}{s-m^2+i\epsilon}+\frac{1}{t-m^2}\right),$$

where the diagram on the left is the s-channel diagram and the one on the right the t-channel.

The lowest order differential cross section then reads

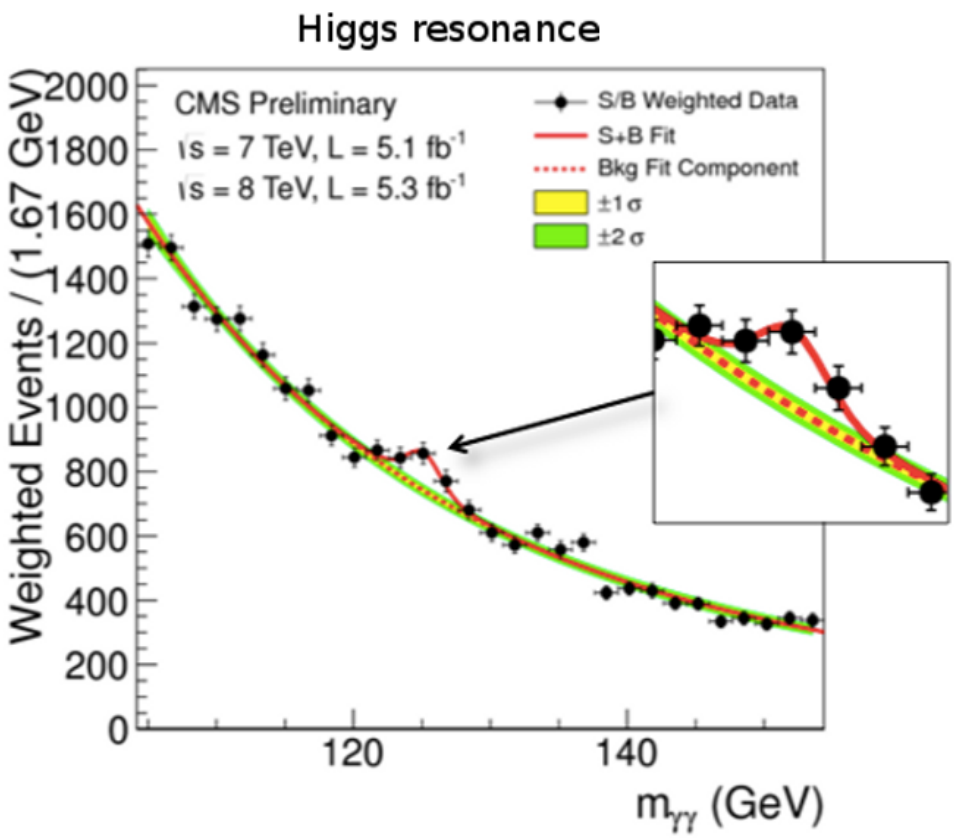
$$\frac{d\sigma^{LO}}{d\Omega} \overset{p.\,65}{=\!=} \frac{\left|\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2)\big)\right|^2}{64\pi^2 s}\,\Theta(\sqrt{s}-2M)\,.$$

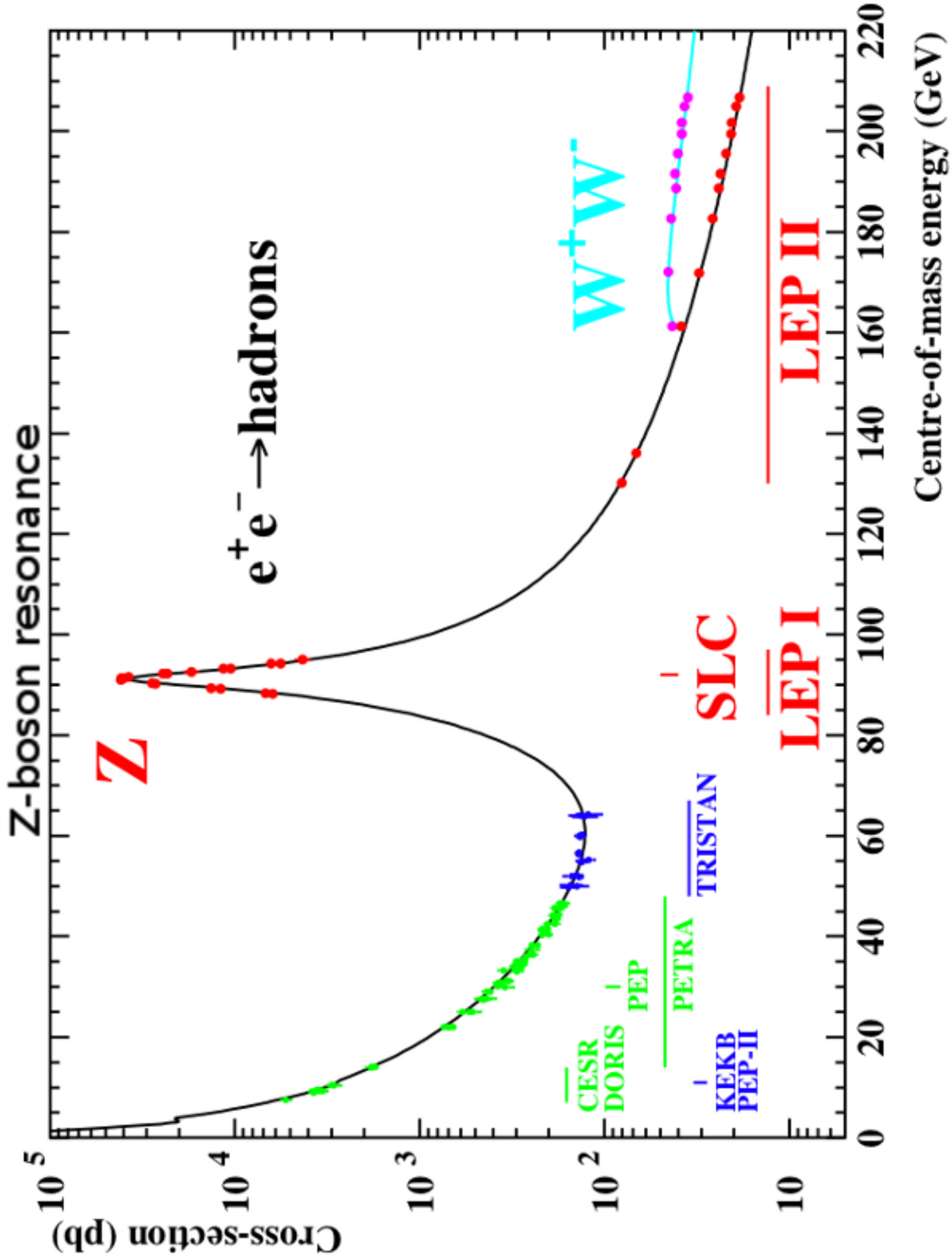
A remarkable feature of this differential cross section originates from the s-channel contribution: it has a singularity at $s = m^2$, i.e. when the intermediate ϕ-particle goes on-shell. This can occur if $m > 2M$, corresponding to an unstable ϕ-particle (see part a). For this reason we kept the $i\epsilon$ imaginary part in the propagator. Actually, when treated correctly this instability adds a finite imaginary piece to the denominator, resulting in a (finite) resonance. The occurrence of such resonances allows to discover new particles! A beautiful example of a recent discovery via a resonance is the discovery of the Higgs boson in 2012 (see attached figure). In the '90s of the last century the resonance of the Z-boson was studied in great detail at LEP (CERN, Geneva) and SLC (SLAC, Stanford). For instance this has led to the conclusion that there are three generations of light neutrinos (see attached plots).

By the way, from $i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2)\big)$ it trivially follows that (see also Ex. 10)

$$V_{\bar{\psi}\psi}(r) \;=\; V_{\psi\psi}(r) \;=\; -\,\frac{(g/2M)^2}{4\pi r}\,e^{-mr}\,,$$

which is a consequence of the invariance of the scalar Yukawa theory under crossing.


Higgs resonance
CMS Preliminary
√s = 7 TeV, L = 5.1 fb-1
√s = 8 TeV, L = 5.3 fb-1
S/B Weighted Data
S+B Fit
Bkg Fit Component
±1 σ
±2 σ
Weighted Events / (1.67 GeV)
0
200
400
600
800
1000
1200
1400
1600
1800
2000
120
140
mγγ (GeV)


Z-boson resonance
Z
e+e− → hadrons
W+W−
Cross-section (pb)
Centre-of-mass energy (GeV)
CESR
DORIS
PEP
PETRA
KEKB
PEP-II
TRISTAN
SLC
LEP I
LEP II

Number of light neutrino generations: $N_\nu = 2.984 \pm 0.008$