Ex. 15) Spin-½ particles in a cube with edges L $\Rightarrow$ volume $V = L^3$, surface area $S = 6L^2$.

Energy eigenvalues: $E_\nu = \dfrac{\hbar^2 \pi^2 \nu^2}{2mL^2}$, $\nu^2 = \nu_x^2 + \nu_y^2 + \nu_z^2$ $\quad (\nu_{x,y,z} = 1, 2, \cdots)$.

Energy eigenfunctions: $\psi_{\vec{\nu}, m_s}(\vec{r}, \sigma) = \left(\dfrac{2}{L}\right)^{3/2} \sin\!\left(\dfrac{\nu_x \pi}{L} x\right) \sin\!\left(\dfrac{\nu_y \pi}{L} y\right) \sin\!\left(\dfrac{\nu_z \pi}{L} z\right) \chi_{\frac{1}{2}, m_s}$ $\quad (m_s = \pm \tfrac{1}{2})$

Consequence: in $\vec{\nu}$-space a unit cube contains precisely one spatial quantum state, provided that $\nu_{x,y,z} > 0$ $\Rightarrow$ $2s+1 = 2$ fully specified quantum states including spin.

Finite-size effect 1: the energy eigenvalues corresponding to $\nu_x = 0 \vee \nu_y = 0 \vee \nu_z = 0$ should be excluded.

see p. 71

(i) According to the partition of $\vec{\nu}$-space introduced in the lecture notes, the number of quantum states with the length of the wave vector smaller than $K = \pi|\vec{\nu}|/L \equiv \pi\nu/L$ is given by $N(K) \xrightarrow[\text{limit}]{\text{cont.}} 2 * [$ volume of a sphere with radius ν restricted to the octant $\nu_{x,y,z} > 0$ $- \frac{1}{2} *$ intersection surface between sphere and octant $]$

(three quarter circles)

$$\Rightarrow N(K) \simeq 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi\nu^3 - 2 \cdot \frac{1}{2} \cdot 3 \cdot \frac{1}{4} \cdot \pi\nu^2 = \frac{1}{3}\pi\nu^3 - \frac{3}{4}\pi\nu^2 = \frac{K^3 L^3}{3\pi^2} - \frac{3K^2L^2}{4\pi} = \frac{V}{3\pi^2}\left(K^3 - \frac{3\pi S}{8V} K^2\right).$$

(ii) Density of states: $D(E_{kin}) = \dfrac{dK}{dE_{kin}} D(K) \xrightarrow[\text{(i)}]{D(K) = dN(K)/dK} \left(\dfrac{dK}{dE_{kin}}\right) \left[\dfrac{V}{\pi^2} K^2 - \dfrac{S}{4\pi} K\right]$

$$\xrightarrow{K = \sqrt{2mE_{kin}/\hbar^2}} \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{1/2} E_{kin}^{-1/2} \left[\frac{V}{\pi^2}\left(\frac{2m}{\hbar^2}\right) E_{kin} - \frac{S}{4\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} E_{kin}^{1/2}\right] = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_{kin}^{1/2} \Big[1 - \underbrace{\frac{\pi S}{4V}\left(\frac{2mE_{kin}}{\hbar^2}\right)^{-1/2}}_{S\lambda(E_{kin})/8V \ll 1 \text{ if } L \gg \lambda(E_{kin})}\Big],$$

which actually holds for arbitrary V and S.

(iii) Now we consider a Fermi gas of such particles. We know that the density of states for given E_{kin} is lowered by a constant $* S$ if finite-size effects (surface) are taken into account $\Rightarrow$ in the ground state of the Fermi gas higher energy levels will have to be filled as compared to a situation without surface effects. Since $D(E_{kin})$ is decreased for all values of the kinetic energy, also $E^{T=0}_{tot,kin}$ will increase!

For a sphere S/V is minimal and therefore the lowest value for the ground-state energy of the Fermi gas for given V is obtained $\Rightarrow$ nuclei are preferably spherical!

Breaking up the volume will cost surface energy, since the volume remains the same whereas the partition wall will increase the amount of surface!

$\hookrightarrow$ nuclei will not break up into "droplets"!

Finite-size effect 2: $\psi_{\vec{\nu},m_s}(\vec{r},\sigma) = 0$ on the edge of the cube.

(iv) Consider $\psi_{\vec{\nu},m_s}(\vec{r},\sigma)$ as a function of x in the vicinity of $x=0$ and take $\nu_x = n_F$:
$|\psi_{\vec{\nu},m_s}(\vec{r},\sigma)| \propto |\sin(\frac{n_F\pi}{L}x)|$, which will become maximal for the first time when $n_F \pi x/L = \pi/2 \Rightarrow \boxed{x_{1st\,max} = \frac{L}{2n_F} = \frac{1}{4} * \text{wavelength}}$.

Ex. 16) Thomas-Fermi model for heavy nuclei: two independent $T=0$ Fermi gases (Z protons, $A-Z$ neutrons, with $M_p \approx M_n$) bound to a sphere with radius R by a constant potential $V = -V_0 < 0$ inside the sphere.

Uniform particle density: $\rho_N = \frac{3A}{4\pi R^3} = \frac{3}{4\pi r_0^3} \approx 0.17 \times 10^{45}$ nucleons/m^3

$\Rightarrow \rho_N^{(p)} = \frac{2Z}{A}(\rho_N/2)$ for the protons, $\rho_N^{(n)} = \frac{2A-2Z}{A}(\rho_N/2)$ for the neutrons.
If $Z = A/2$, then $E_F = \frac{\hbar^2}{2M_p}(3\pi^2\rho_N/2)^{2/3}$ is the Fermi energy for both gases.

(i) Total kinetic energy: $E^{T=0}_{tot,kin} = \frac{3}{5} Z E_F^{(p)} + \frac{3}{5}(A-Z)E_F^{(n)}$, with $E_F^{(p/n)} = \left(\frac{\rho_N^{(p/n)}}{\rho_N/2}\right)^{2/3} E_F$

$\Rightarrow E^{T=0}_{tot,kin} = \frac{3}{5} E_F \left(\frac{A}{2}\right)\left[x^{5/3} + (2-x)^{5/3}\right]$, using $x = 2Z/A$.

Minimizing for fixed A: $dE^{T=0}_{tot,kin}/dx = 0 \Rightarrow x^{2/3} - (2-x)^{2/3} = 0 \Rightarrow x = 1$, i.e. $Z = A/2$.

(ii) Take $x = 1-\lambda$, $0 < \lambda \ll 1 \Rightarrow E^{T=0}_{tot,kin} = \frac{3}{10} AE_F\left[(1-\lambda)^{5/3} + (1+\lambda)^{5/3}\right] \overset{\lambda \ll 1}{\approx} \underbrace{\frac{3}{5}AE_F(1 + \frac{5}{9}\lambda^2 + \cdots)}$

$$\approx \frac{3}{5}AE_F + \frac{(A-2Z)^2}{3A}E_F$$

Hence: asymmetry coeff. $\frac{1}{3}E_F \approx 13$ MeV vs 23.2 MeV measured ← ~50% explained

(Coulomb energy of spherical nucleus)

(iii) Minimizing $E^{T=0}_{tot} = E^{T=0}_{tot,kin} - AV_0 + \Delta V_{Coul} \overset{(i)}{=} \frac{3}{10}AE_F(x^{5/3} + (2-x)^{5/3}) - AV_0 + \frac{3\alpha\hbar c}{20 r_0}A^{5/3}x^2$

see §2.5.5 for E_{grav}

$\xrightarrow{A \text{ fixed}} dE^{T=0}_{tot}/dx = 0$, i.e. $\boxed{x^{2/3} - (2-x)^{2/3} + \frac{3\alpha\hbar c}{5 r_0 E_F}A^{2/3}x = 0}$.

$Z < A-Z$: less protons than neutrons

Consequence: minimum for $x = 2Z/A < 1$, stronger deviation from $x=1$ for larger A!

(iv) Quantum mechanical Coulomb corrections due to proton pair interactions depend on the total spin of the proton pair:

spin triplet ($S=1$) $\Rightarrow$ spatially antisymmetric, more distance between the protons $\Rightarrow$ weaker Coulomb effects,

spin singlet ($S=0$) $\Rightarrow$ spatially symmetric, less distance between the protons $\Rightarrow$ stronger Coulomb effects.

(compared to the classical approach)

Ex. 17) Free spin-1/2 particles inside a macroscopic 3-dimensional enclosure with fixed edges $L = V^{1/3}$ and impenetrable walls. Number of particles: $N \gg 1$, const.

1-particle energy eigenvalues in the ultra-relativistic limit:

$$\underbrace{E_\nu = \frac{\hbar \pi c}{L}}\,\nu\,,\quad \text{with } \nu = \sqrt{\nu_x^2 + \nu_y^2 + \nu_z^2} \quad (\nu_{x,y,z} = 1,2,\ldots).$$

$\Longrightarrow$ $E(k) = \hbar c k$ in terms of the quantized wave vectors $\vec{k} = \pi \vec{\nu}/L$

In the lecture notes it was derived that $D(k) = Vk^2/\pi^2$

$$\Rightarrow \text{density of states: } D(E_{kin}) = \left(\frac{dk}{dE_{kin}}\right) D(k) \overset{k = E_{kin}/\hbar c}{=} \frac{V}{\pi^2}\frac{1}{\hbar c}\left(\frac{E_{kin}}{\hbar c}\right)^2 = \frac{V E_{kin}^2}{\pi^2 \hbar^3 c^3}.$$

Fermi gas at $T=0$: all 1-particle states are occupied up to the Fermi energy

$$N = \int_0^{E_F} dE_{kin}\, D(E_{kin}) = \frac{V E_F^3}{3\pi^2(\hbar c)^3} \Rightarrow \text{Fermi energy: } E_F = \hbar c\,(3\pi^2 \underbrace{N/V}_{\rho_N})^{1/3},$$

$$\bar{E}_{kin} \overset{T=0}{=} \frac{E_{kin,tot}}{N} = \frac{1}{N}\int_0^{E_F} dE_{kin}\, E_{kin}\, D(E_{kin}) = \frac{1}{N}\frac{V E_F^4}{4\pi^2(\hbar c)^3} = \boxed{\frac{3}{4}E_F = \bar{E}_{kin}} \leftarrow \text{average kin. energy per particle}$$

$$\Rightarrow \text{gas pressure } P \overset{T=0}{=} -\left(\frac{\partial E_{kin,tot}}{\partial V}\right)_N = -\frac{3}{4}N\left(\frac{\partial E_F}{\partial V}\right)_N = \frac{1}{4}N\frac{E_F}{V} = \boxed{\frac{1}{4}\rho_N E_F = P}.$$

This type of Fermi gas is relevant for the thermodynamic treatment of neutrinos at low temperatures or electrons/neutrons inside heavy imploded stars.

Ex. 18) Massive white dwarf, model 1: ultra relativistic electron gas at $T=0$ inside a spherical volume $\Rightarrow \rho_N^{(e)} = \dfrac{N_e}{\frac{4}{3}\pi R^3} = \dfrac{(M/Am_p)Z}{\frac{4}{3}\pi R^3}$.

(i) Total energy of the star (see lecture notes): $E_T = N_e \bar{E}_{kin} - \dfrac{3}{5}\dfrac{G_N M^2}{R}$ — kinetic energy; grav. energy; M: mass star; R: radius star

$$\overset{\text{ex. 17}}{=} \frac{MZ}{Am_p}\,\frac{3}{4}\hbar c\left(\frac{3\pi^2 (MZ/Am_p)}{\frac{4}{3}\pi R^3}\right)^{1/3} - \frac{3}{5}\frac{G_N M^2}{R} = b\,\frac{M^{4/3}}{R} - \frac{3}{5}\frac{G_N M^2}{R} = E_T,$$

$$\text{with } b = \frac{3}{4}\left(9\pi/4\right)^{1/3}\hbar c\left(\frac{Z}{Am_p}\right)^{4/3} \overset{Z=A/2}{=} 9.096 \times 10^{9}\ \mathrm{J\,m\,kg^{-4/3}}.$$

(ii) Equilibrium: unlike the NR case discussed in the lecture notes, this time there is no minimum for E_T as function of the radius R.

$E_T < 0$: kinetic energy < |grav. energy|

$\Rightarrow$ star continues to implode, since smaller R lowers the energy and the increasing kinetic energy remains ultra relativistic.

$E_T > 0$: kinetic energy > |grav. energy|

$\Rightarrow$ star expands, since this time larger R lowers the energy.

However, the decreasing kinetic energy gradually becomes less relativistic $\Rightarrow$ eventually the star will reach a stable equilibrium value for R, since a non-rel. white dwarf has a stable minimum for E_T!

So, there is an upper bound on the mass of a stable white dwarf:

$$bM^{4/3} \geq \frac{3}{5}G_N M^2 \Rightarrow M \leq M_c = (5b/3G_N)^{3/2} \overset{Z=A/2}{\approx} \frac{15\sqrt{5\pi}}{64}\frac{(\hbar c/G_N)^{3/2}}{M_p^2} \overset{\text{hint}}{=} 1.72\,M_\odot .$$

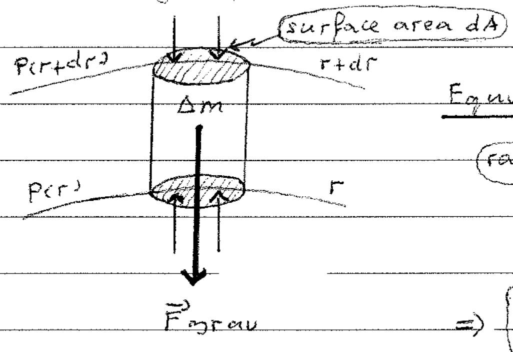
Massive white dwarf, model 2: refining model 1.

defines the radius of the star

Mass density profile $\rho_M(r) = \frac{AM_p}{Z}\rho_N^{(e)}(r) \equiv \rho_c\,\Theta^3(r)$, with $\Theta(0) \equiv 1$ and $\Theta(R) = 0$

polytropic eqn. of state

$\Rightarrow$ electron gas pressure $P(r) \overset{\text{ex. 17}}{=} \frac{1}{4}\hbar c\,(3\pi^2)^{1/3}\left(\frac{Z}{AM_p}\right)^{4/3}\rho_M^{4/3}(r) \equiv K\rho_M^{4/3}(r) = K\rho_c^{4/3}\,\Theta^4(r)$.



(iii) Equilibrium: $F_r = \Delta m\frac{d^2r}{dt^2} = \overbrace{-\frac{G_N\Delta m\, m(r)}{r^2}}^{\text{gravity}} + dA\,\overbrace{\underbrace{(P(r) - P(r+dr))}_{-P'(r)\,dr}}^{\text{gas pressure}}$

radial force

$$= dA\,dr\left(-\frac{G_N\, m(r)\rho_M(r)}{r^2} - P'(r)\right) = 0$$

$$\Rightarrow \boxed{m(r) = -\frac{r^2}{G_N\rho_M(r)}P'(r)}$$

mass of the star inside a radial distance r

Plugging in $P(r) = K\rho_M^{4/3}(r) = K\rho_c^{4/3}\Theta^4(r)$ results in

$$m(r) = -\frac{r^2}{G_N\rho_c\Theta^3(r)}\,4K\rho_c^{4/3}\,\Theta^3(r)\frac{d\Theta(r)}{dr} = -\frac{4K}{G_N}\rho_c^{1/3}\,r^2\frac{d\Theta(r)}{dr}.$$

(iv) Definition: $\rho_c\,\Theta^3(r) = \rho_M(r) \equiv \frac{dm(r)/dr}{4\pi r^2} \overset{(iii)}{=} -\frac{K}{\pi G_N}\rho_c^{1/3}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Theta(r)}{dr}\right)$

$$\xRightarrow{r = \rho_c^{-1/3}\left(\frac{K}{\pi G_N}\right)^{1/2}\xi} \boxed{\Theta^3 + \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = 0}$$

Lane-Emden eqn. for $n=3$, free of star-specific parameters

(v) The desired solution is fixed by the boundary conditions $\Theta(0) = 1$, $\Theta'(0) = 0$.

equil.: $dP/dr|_{r=0} = 0$

Radius R: $\xi = \xi_R = 6.897$, for which $\Theta = 0$, $\xi^2 d\Theta/d\xi = -2.018$.

Stellar mass: $M_c \equiv m(R) \overset{(iii)}{=} -4\pi\frac{K}{\pi G_N}\rho_c^{1/3}R^2\frac{d\Theta}{dr}\Big|_{r=R} \overset{r\to\xi}{=} -4\pi\left(\frac{K}{\pi G_N}\right)^{3/2}\xi^2\frac{d\Theta}{d\xi}\Big|_{\xi=\xi_R}$

$$= 8.072\,\pi\left(\frac{K}{\pi G_N}\right)^{3/2} \overset{Z=A/2}{\approx} 8.072\frac{\sqrt{3\pi}}{32}\frac{(\hbar c/G_N)^{3/2}}{M_p^2} \overset{\text{hint}}{=} 1.44\,M_\odot .$$

This is the so-called Chandrasekhar limit, which places an upper limit on the mass of a stable (non-rotating) white dwarf!

the ultra-rel. assumption has stretched it to the limit