Discussion summary:

(a) **Non-relativistic Coulomb potential**: in analogy to exercise 9(c) the lowest-order amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2)\big)$ for particle-particle scattering in scalar QED reads

$$(-ie)^2 \left(\frac{-i\eta_{\mu\nu}}{(k_A-p_2)^2+i\epsilon}\,(k_A+p_2)^\mu(k_B+p_1)^\nu + \frac{-i\eta_{\mu\nu}}{(k_A-p_1)^2+i\epsilon}\,(k_A+p_1)^\mu(k_B+p_2)^\nu \right),$$

whereas for antiparticle-particle scattering $i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2)\big)$ is given by

$$(-ie)^2 \left(\frac{-i\eta_{\mu\nu}}{(k_A+k_B)^2+i\epsilon}\,(k_A-k_B)^\mu(p_1-p_2)^\nu + \frac{-i\eta_{\mu\nu}}{(k_A-p_1)^2+i\epsilon}\,(k_A+p_1)^\mu(-p_2-k_B)^\nu \right).$$

The latter amplitude is obtained from the former one by means of crossing, i.e. $k_B \leftrightarrow -p_2$.

In the non-relativistic limit one obtains the approximations

(i) $(k_A+p_2)\cdot(k_B+p_1) = (k_A+p_2)^0(k_B+p_1)^0 - (\vec{k}_A+\vec{p}_2)\cdot(\vec{k}_B+\vec{p}_1) \approx (k_A+p_2)^0(k_B+p_1)^0$, since in the non-relativistic limit $E \approx M \gg |\vec{p}\,|$. The initial-state particles have the same mass as the final-state particles, hence $(k_A+p_2)^0(k_B+p_1)^0 \approx (2M)^2$. In the same manner:

(ii) $(k_A+p_1)\cdot(k_B+p_2) \approx (k_A+p_1)^0(k_B+p_2)^0 \approx (2M)^2$

$$\Rightarrow \quad i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2)\big) \approx \frac{i\,(2Me)^2}{(k_A-p_2)^2+i\epsilon} + \frac{i\,(2Me)^2}{(k_A-p_1)^2+i\epsilon}.$$

For antiparticle-particle scattering it follows that:

(i) $(k_A-k_B)\cdot(p_1-p_2) \approx 0$ at leading order and the first non-vanishing order yields
$(k_A-k_B)\cdot(p_1-p_2) \approx 0 - (\vec{k}_A-\vec{k}_B)\cdot(\vec{p}_1-\vec{p}_2)$;

(ii) $(k_A+p_1)\cdot(-p_2-k_B) \approx -(2M)^2$ [having the opposite sign due to crossing!].

Hence, numerator (i) is supressed by $\mathcal{O}(\vec{k}\cdot\vec{p}/M^2)$ compared to numerator (ii). This suppression is further enhanced by the two denominators: $(k_A+k_B)^2 \approx (2M)^2 \gg (k_A-p_1)^2$. The second term therefore dominates the matrix element:

$$i\mathcal{M}^{LO}\big(\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2)\big) \approx \frac{-\,i\,(2Me)^2}{(k_A-p_1)^2+i\epsilon}.$$

Comparing these two equations for the matrix elements in the non-relativistic limit to the one obtained for the scalar Yukawa theory on page 55 of the lecture notes, one concludes that we have to substitute g by $2Me$ and replace the mass of the exchanged virtual particle by 0. In this way we obtain the well-known Coulomb potential

$$V_{\psi\psi}(r) = +\frac{e^2}{4\pi r} = -\,V_{\bar{\psi}\psi}(r).$$

In (scalar) QED the interaction between particles is repulsive, which is caused by the $-\,\eta_{\mu\nu}$ propagator factor, whereas the interaction between particles and antiparticles is attractive, which is caused by crossing. The bosonic particle being exchanged here as force carrier (photon) is a spin-1 particle. Spin-1 exchange allows both attractive and repulsive interactions!

With respect to a spin-1-exchange interaction, particles and antiparticles have opposite charge: equal charges repel each other, opposite charges attract each other.

Remark: as you can see from the Feynman rule involving one photon and two scalar particles, the photon couples to a conserved charge current with one Lorentz index μ*. Upon contraction with the photon momentum* $(p-p')_\mu$ *one indeed obtains* $-ie\,(p^2-p'^2)=0$ *for on-shell scalar particles, as required for a conserved current.*

(b) *Next we consider the coupling between the spin-2 graviton and scalar particles. In this case the graviton couples to the conserved energy-momentum tensor for the scalar matter, which has two Lorentz indices* μ *and* ν*. As can be checked, contracting the given Feynman rule by either* $(p-p')_\mu$ *or* $(p-p')_\nu$ *indeed yields 0 for on-shell scalar particles.*

Non-relativistic gravitational potential: the interactions mediated by spin-2 gravitons can now be analyzed in the same way as we previously did for scalar QED. For particle-particle scattering the lowest-order amplitude $i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B)\to\psi(p_1)\psi(p_2)\big)$ reads according to exercise 9(c)

$$\big(i\sqrt{8\pi G}\,\big)^2\,\frac{\frac{i}{2}\,(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta}-\eta_{\alpha\beta}\eta_{\mu\nu})}{(k_A-p_1)^2+i\epsilon}\,T^{\mu\nu}(k_A,p_1)T^{\alpha\beta}(k_B,p_2)\;+\;(p_1\leftrightarrow p_2)\,.$$

In the non-relativistic limit $q\cdot r\approx M^2$ for all contractions involving the initial- and final-state momenta. Thus to leading order the amplitude simplifies considerably:

$$i\mathcal{M}^{LO}\big(\psi(k_A)\psi(k_B)\to\psi(p_1)\psi(p_2)\big)\;\approx\;\frac{-\,i\,(16\pi GM^4)}{(k_A-p_1)^2+i\epsilon}+\frac{-\,i\,(16\pi GM^4)}{(k_A-p_2)^2+i\epsilon}\,.$$

Therefore, by analogy to the Yukawa case the associated non-relativistic potential is given by

$$V_{\psi\psi}(r)=-\,\frac{GM^2}{r}\,,$$

i.e. Newton's gravitational potential.

The particle-antiparticle scattering amplitude is again obtained by crossing:

$$\big(i\sqrt{8\pi G}\,\big)^2\,\frac{\frac{i}{2}\,(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta}-\eta_{\alpha\beta}\eta_{\mu\nu})}{(k_A-p_1)^2+i\epsilon}\,T^{\mu\nu}(k_A,p_1)T^{\alpha\beta}(-p_2,-k_B)\;+\;(p_1\leftrightarrow -k_B)\,.$$

The second term is suppressed with respect to the first one, since $(k_A+k_B)^2\gg(k_A-p_1)^2$, but this time $T^{\alpha\beta}(-p_2,-k_B)=T^{\alpha\beta}(k_B,p_2)$ as opposed to $J^\nu(-p_2,-k_B)=-J^\nu(k_B,p_2)$ in the scalar QED case. So, just like in the case of Yukawa spin-0-exchange the relevant vertex is invariant under crossing, and the particle-antiparticle amplitude is the same as for particle-particle scattering. Hence, the non-relativistic potentials

$$V_{\bar{\psi}\psi}(r) \,=\, V_{\psi\psi}(r) \,=\, -\,\frac{GM^2}{r}$$

are identical and attractive, as expected for a gravitational potential.

To summarize: for force carriers with even spin the relevant interaction vertex is a tensor with an even number of Lorentz indices. If the particles that feel the force are scalar, this implies that the vertex is a sum of products of an even number of momenta. Such a vertex is invariant under crossing of the momenta at that vertex (switching from particles to antiparticles) and the corresponding interaction is universally attractive. For force carriers with odd spin, the number of indices/momenta of the interaction vertex is odd, and one obtains a repulsive/attractive force for particle-particle/antiparticle-particle interactions.

Justification for the given graviton − scalar interaction vertex: for the free complex scalar field the energy-momentum tensor can be calculated straightforwardly to be (see pages 8 and 12 of the lecture notes):

$$\hat{T}^{\mu\nu} \,=\, \partial^\mu\hat{\phi}^\dagger\partial^\nu\hat{\phi} + \partial^\nu\hat{\phi}^\dagger\partial^\mu\hat{\phi} - \eta^{\mu\nu}\left(\partial_\rho\hat{\phi}^\dagger\partial^\rho\hat{\phi} - M^2\hat{\phi}^\dagger\hat{\phi}\right)$$

with

$$\hat{\phi}(\vec{x}\,) \,=\, \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}}\,e^{-ip\cdot x} + \hat{b}^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)_{p^0=\omega_{\vec{p}}},$$

and the hermitean conjugate for $\hat{\phi}^\dagger(x)$. The free 1-particle momentum states are generically given by $|\vec{k}\,\rangle = \sqrt{2\omega_{\vec{k}}}\,\hat{a}^\dagger_{\vec{k}}|0\rangle$. This means that in order to calculate $\langle\vec{p}\,'|\hat{T}^{\mu\nu}(x)|\vec{p}\,\rangle$ for $\vec{p}\neq\vec{p}\,'$, with $|\vec{p}\,'\rangle$ and $|\vec{p}\,\rangle$ being different ψ-particle momentum states, one needs the following matrix elements:

$$\langle 0|\hat{a}_{\vec{p}\,'}\hat{a}^\dagger_{\vec{q}}\,\hat{a}_{\vec{q}\,'}\hat{a}^\dagger_{\vec{p}}|0\rangle \,=\, (2\pi)^6\delta(\vec{q}-\vec{p}\,')\,\delta(\vec{q}\,'-\vec{p}\,)\,,$$

$$\langle 0|\hat{a}_{\vec{p}\,'}\hat{a}^\dagger_{\vec{q}}\,\hat{b}^\dagger_{\vec{q}\,'}\hat{a}^\dagger_{\vec{p}}|0\rangle \,=\, \langle 0|\hat{a}_{\vec{p}\,'}\hat{b}_{\vec{q}}\,\hat{a}_{\vec{q}\,'}\hat{a}^\dagger_{\vec{p}}|0\rangle \,=\, \langle 0|\hat{a}_{\vec{p}\,'}\hat{b}_{\vec{q}}\,\hat{b}^\dagger_{\vec{q}\,'}\hat{a}^\dagger_{\vec{p}}|0\rangle \,=\, 0\,,$$

where the last term equals 0 because $\vec{p}\neq\vec{p}\,'$.

This leads to

$$\langle\vec{p}\,'|\partial^\mu\hat{\phi}^\dagger\partial^\nu\hat{\phi}\,|\vec{p}\,\rangle \,=\, \sqrt{2\omega_{\vec{p}\,'}}\,\frac{1}{\sqrt{2\omega_{\vec{p}\,'}}}\,(ip'^\mu e^{ip'\cdot x})\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,(-ip^\nu e^{-ip\cdot x})\sqrt{2\omega_{\vec{p}}} \,=\, p'^\mu p^\nu e^{i(p'-p)\cdot x}\,,$$

and similarly one obtains $\langle\vec{p}\,'|\partial^\nu\hat{\phi}^\dagger\partial^\mu\hat{\phi}\,|\vec{p}\,\rangle = p'^\nu p^\mu e^{i(p'-p)\cdot x}$, $\langle\vec{p}\,'|\partial_\rho\hat{\phi}^\dagger\partial^\rho\hat{\phi}\,|\vec{p}\,\rangle = p'_\rho p^\rho e^{i(p'-p)\cdot x}$ and $\langle\vec{p}\,'|M^2\hat{\phi}^\dagger\hat{\phi}\,|\vec{p}\,\rangle = M^2 e^{i(p'-p)\cdot x}$.

Putting everything together:

$$\langle\vec{p}\,'|\hat{T}^{\mu\nu}(x)|\vec{p}\,\rangle \,=\, e^{i(p'-p)\cdot x}\left[\,p'^\mu p^\nu + p'^\nu p^\mu - \eta^{\mu\nu}(p'\cdot p - M^2)\,\right] \,\equiv\, e^{i(p'-p)\cdot x}\,T^{\mu\nu}(p',p)\,.$$

For antiparticle states the calculation can be performed in an analogous way.

Rules of thumb for derivative couplings: the previous discussion can be summarized in terms of momentum-space Feynman rules for interactions involving fields with derivatives acting on it. For an incoming particle with momentum q, the field derivative $\partial^\rho \hat{\phi}$ yields a vertex factor $-iq^\rho$. For an outgoing particle with momentum q, the field derivative $\partial^\rho \hat{\phi}^\dagger$ yields a vertex factor $+iq^\rho$. Implementing this in the above-given graviton – scalar interaction instantly reproduces the momentum-space Feynman rule for the graviton – scalar interaction vertex.

In the case of scalar QED something similar happens. As will be explained in chapter 5, the photon field will couple to the conserved Noether current resulting from the global $U(1)$ gauge symmetry of the free theory that describes the considered charged particles (which are scalar in our case). On page 12 of the lecture notes the conserved current for the complex Klein – Gordon theory was found to be

$$i\hat{\phi}(x)\partial^\mu \hat{\phi}^\dagger(x) - i\hat{\phi}^\dagger(x)\partial^\mu \hat{\phi}(x) \ ,$$

which would translate into a momentum-space vertex factor $-(p+p')^\mu$ for an incoming particle with momentum p and an outgoing particle with momentum p'. Up to a charge proportionality factor this indeed coincides with the advertised momentum-space vertex rule for scalar QED.